Deforming Geometric Structures in Low Dimensions

$\mathbb{H}^g\mathbb{E}^l$ - Community Seminar 2025

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What is a Geometry?

think of geometry as a pair (G, X) G acts transitively and strongly effectively on X.

$$p \xrightarrow{g \in G} q$$

$$U \subset X$$

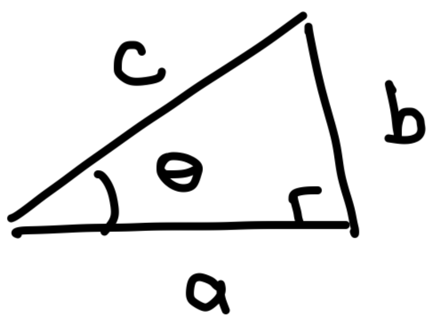
$$\{a\ b\ c\ d\} \xrightarrow{g} \{a\ b\ c\ d\}$$

then $g = 1 \in G$

Euclidean

$$G = \mathrm{Isom}(\mathbb{E}^2)$$

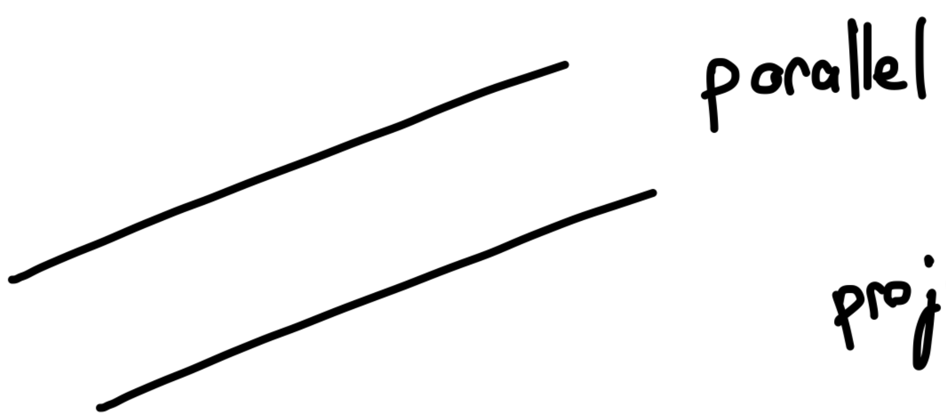
$$X = (\mathbb{E}^2, \|\cdot\|_2)$$

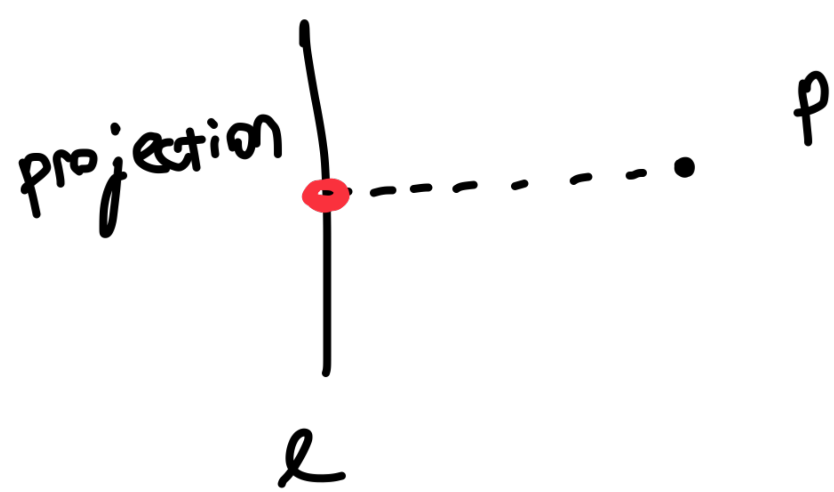


$$a^2 + b^2 = c^2$$

$$\sin\theta = b/c$$

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

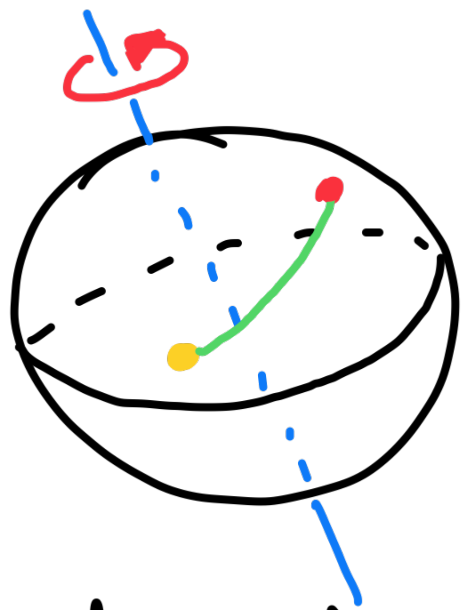




(Spherical)

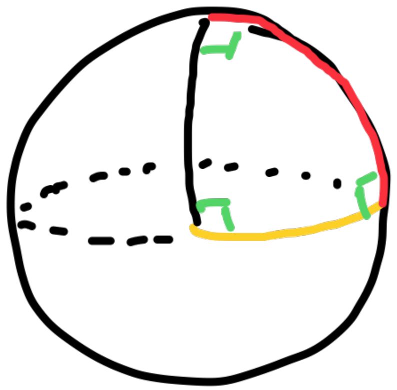
$G = O(3, \mathbb{R})$

$X = S^2$



$\subset \mathbb{R}^3$

parallel out window ¨

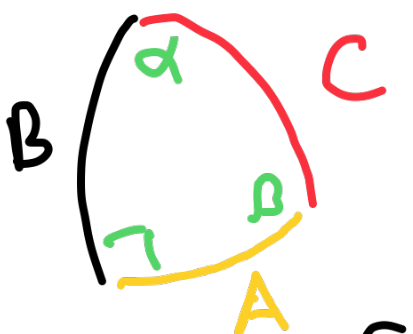


triangles still a thing

angle sums all greater than π

area () $= \frac{\pi}{2}$ more generally Area $= (\sum \text{angles} - \pi)$

point projection gets funky



$\cos(C) = \cos(A)\cos(B)$

Hyperbolic

$G = PGL(2, \mathbb{R})$

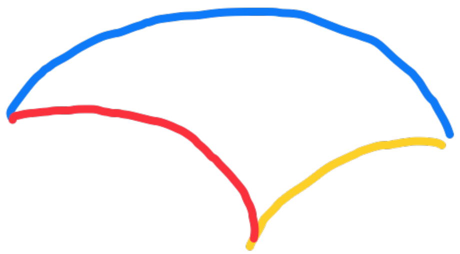
$X = \mathbb{H}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

lines are semi-circles $\perp$ to real axis, or, vertical lines.

$\mathbb{H}^2$

triangle

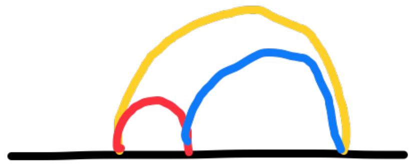


$\partial \mathbb{H}^2 \cong$ 

parallelism makes sense → ultra-parallel &

→ (asymptotically) parallel &

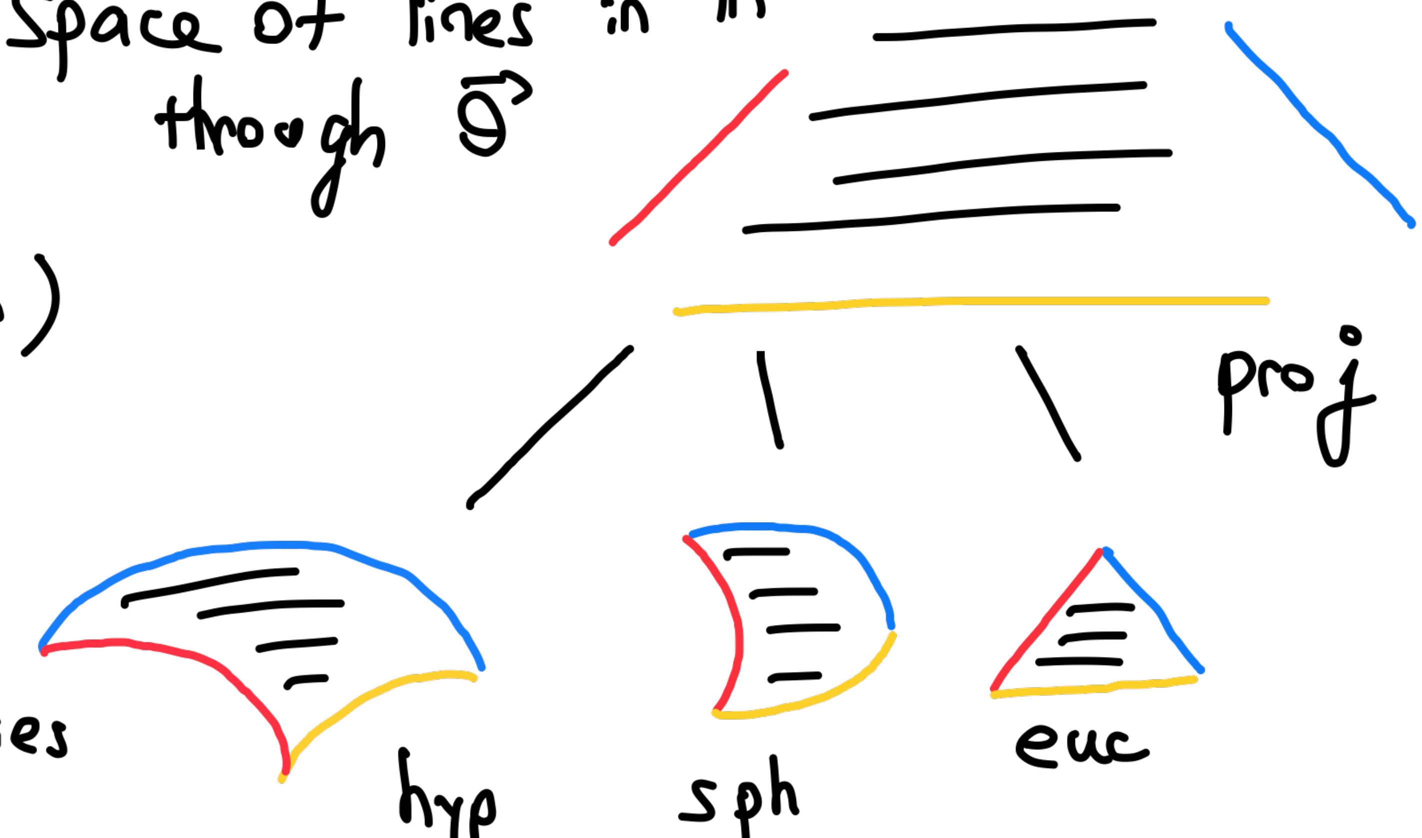
Area of $\Delta = \pi - \sum$ angles

Area () $= \pi$

Projective

$X = \mathbb{RP}^2 =$ space of lines in $\mathbb{R}^3$ through $\vec{0}$

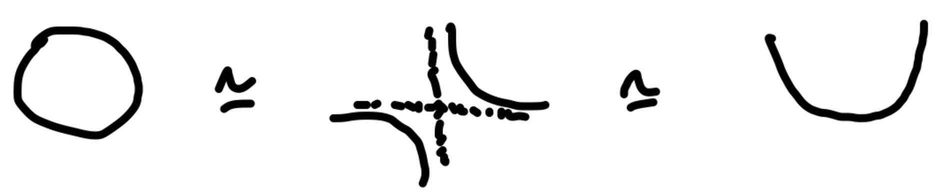
$G = PGL(3, \mathbb{R})$



unites all 3-geometries

however, loses metric in process.

$x^2 + y^2 = 1 \cong x^2 - y^2 = 1 \cong y - x^2 = 1$

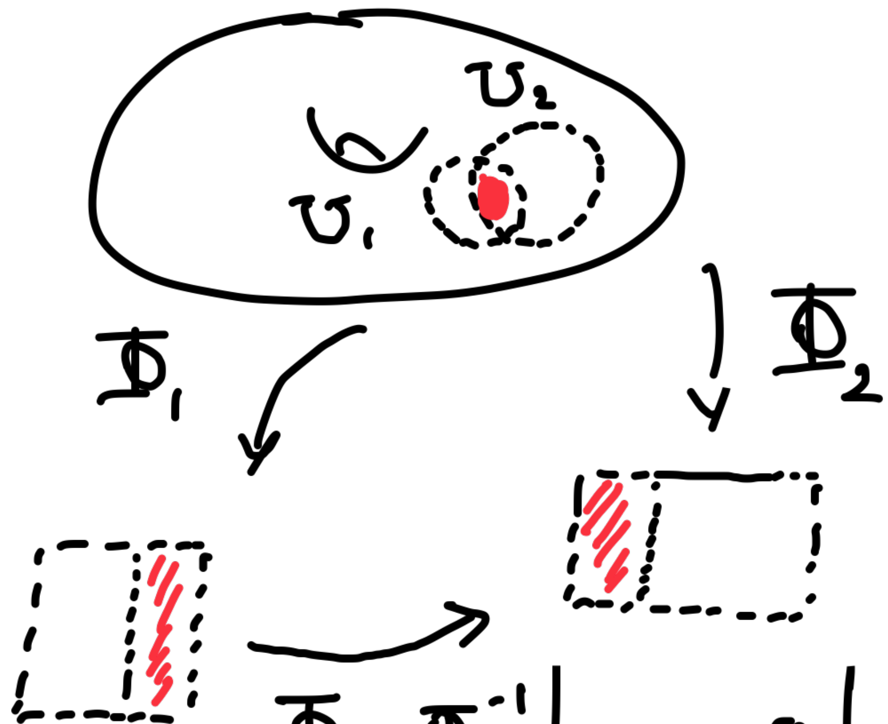


in $\mathbb{RP}^2$

Geometric Structure on Manifold

Let (G, X) be a geometry and M a manifold.

A (G,X)-structure on M is a collection of charts $\overset{\subset M}{U_\alpha} \xrightarrow{\Phi_\alpha} X$ so that locally coordinate changes are symmetries in G.

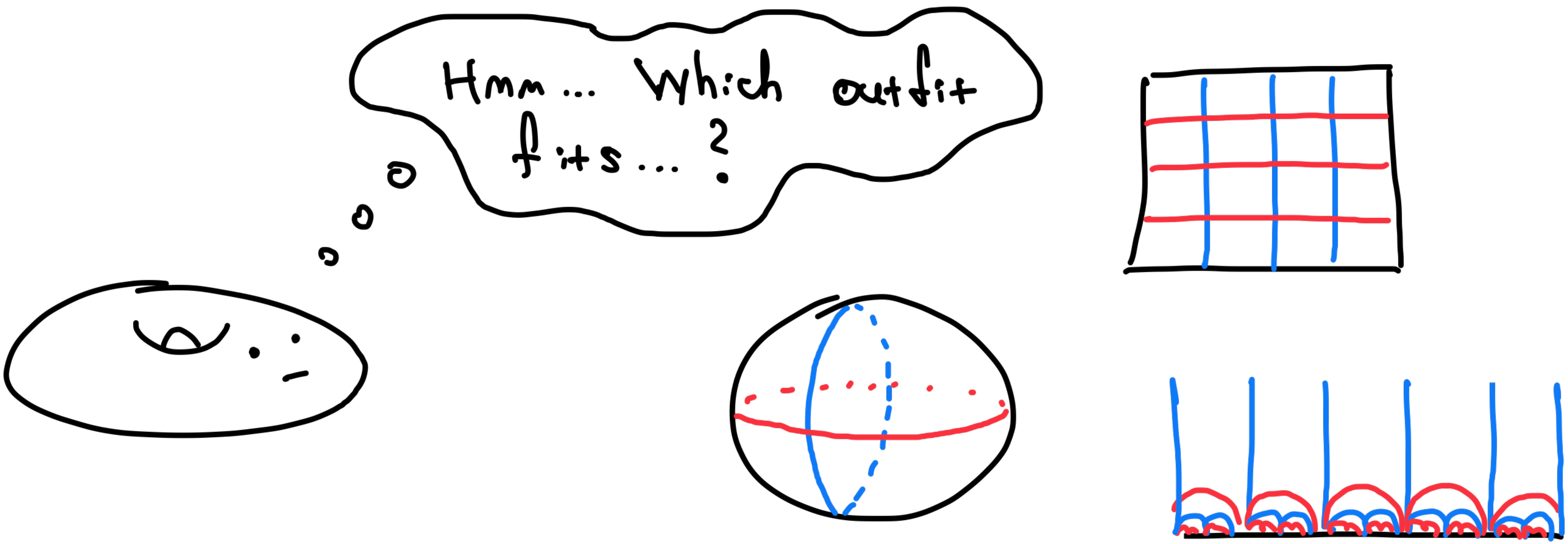


$\Phi_2 \circ \Phi_1^{-1}|_{\blacksquare} = g|_{\blacksquare}$

This means locally M is modelled by the geometry (G, X).

Question One:

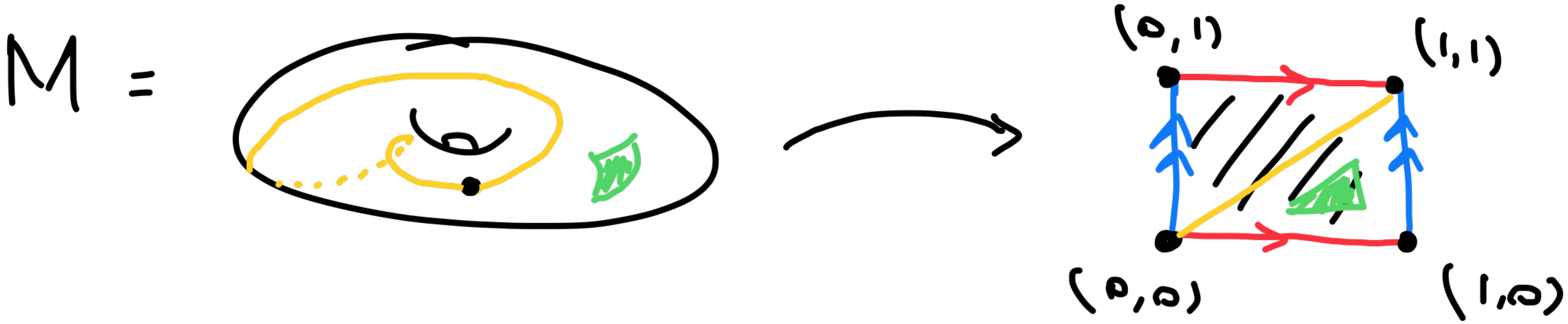
1. Given a manifold M, what type of (G, X) structures can it admit?



Question Two

2. If you can put a (G,X)-structure on M, how many inequivalent ways can you do it?

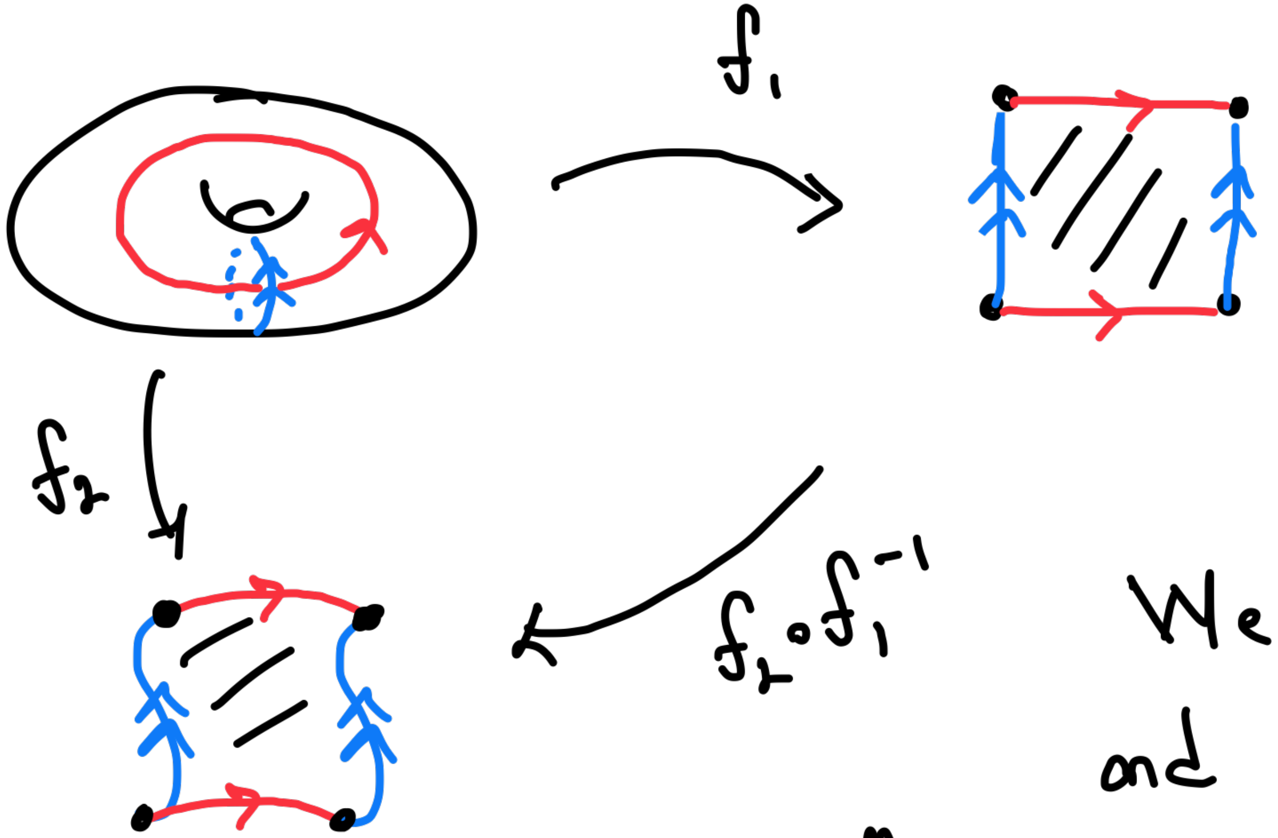
Example



Gives T^2 a euclidean structure. So on T^2 we can define a notion of a <u>line</u>. We can also say when something is a <u>triangle</u> on T^2. Area, length, angles, parallelism, etc are all inherited on our torus.

Example (cont)

Want to consider structures up to some equivalence



f_1 and f_2 defining some Euclidean structure on T^2 (pretty much)

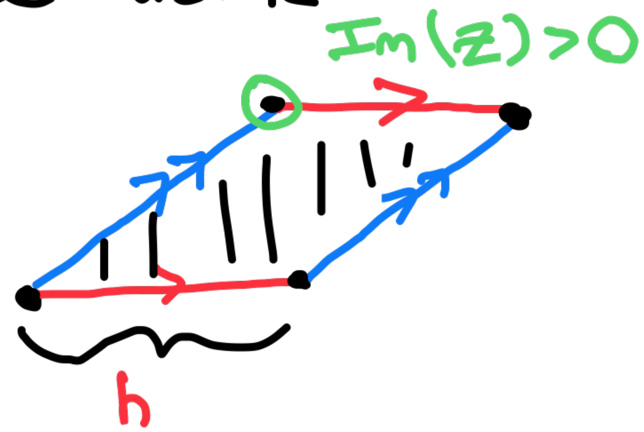
We say $f_1 : T^2 \twoheadrightarrow (S_1, g_1)$ and $f_2 : T^2 \twoheadrightarrow (S_2, g_2)$ are equivalent iff $f_2 \circ f_1^{-1} : (S_1, g_1) \twoheadrightarrow (S_2, g_2)$ isotopic to an isometry

f_1, f_2 called markings

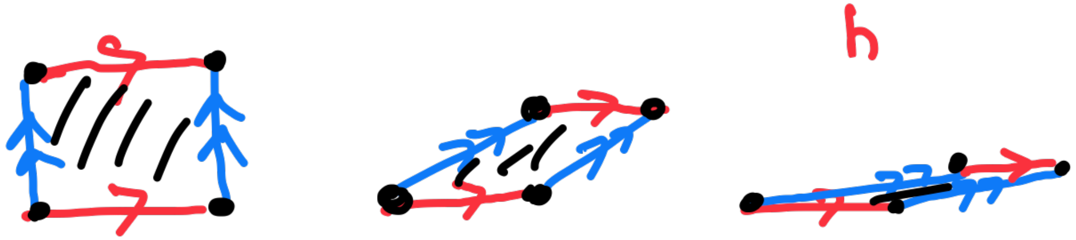
Example continued

Study space of (Marked) (G,X)-structures on manifold up to equivalence, $X(M)$.

For T^2 we can think of space of marked Euclidean structures as space of parallelograms with one edge horizontal and other edge above horizontal axis.



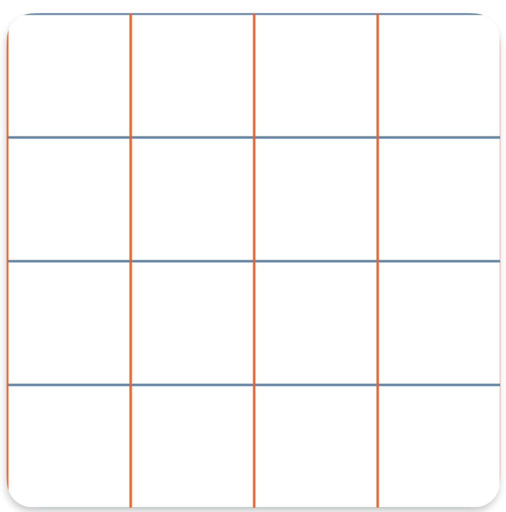
quick dimension count gives 3 real.

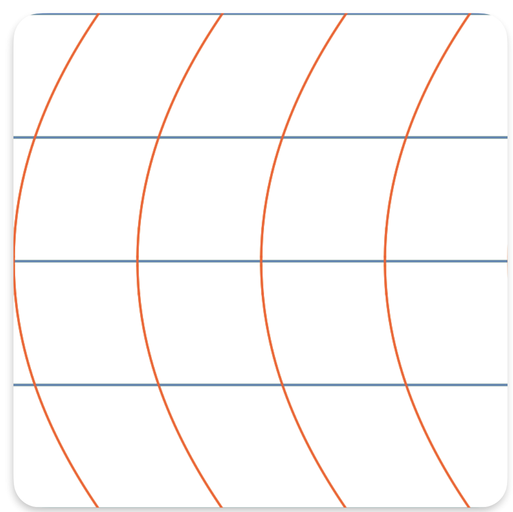


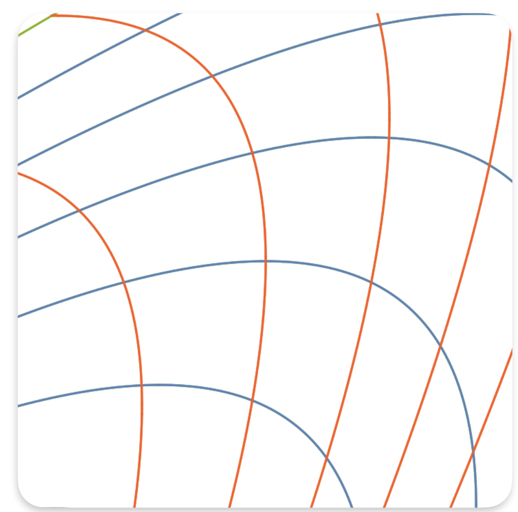
What about non-Euclidean T^2?

A particularly nice feature of the Euclidean tori is they're quotients of $\mathbb{R}^2$ by discrete subgroups of $\mathrm{Isom}(\mathbb{E}^2)$. What about $\mathrm{Aff}(\mathbb{A}^2) \cong GL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

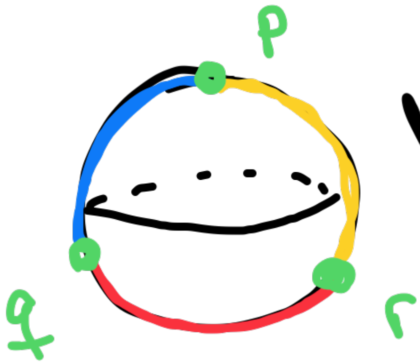
$\mathbb{A}^2_{comp}(T^2) \cong \mathbb{R}^2$ where origin corresponds to all Euclidean structures on T^2. (BG)

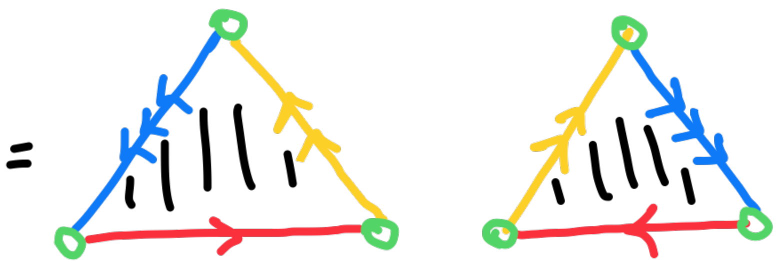


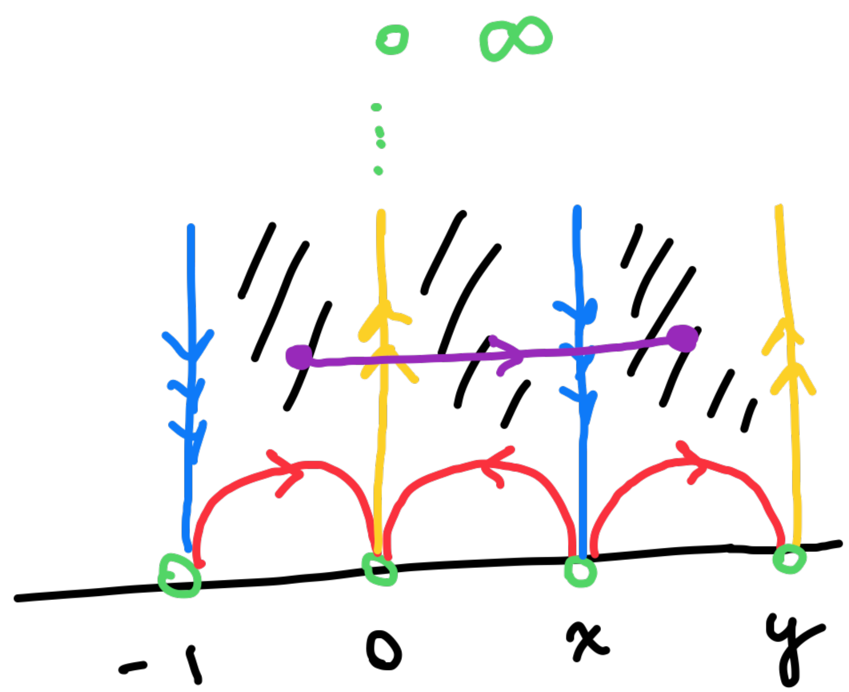




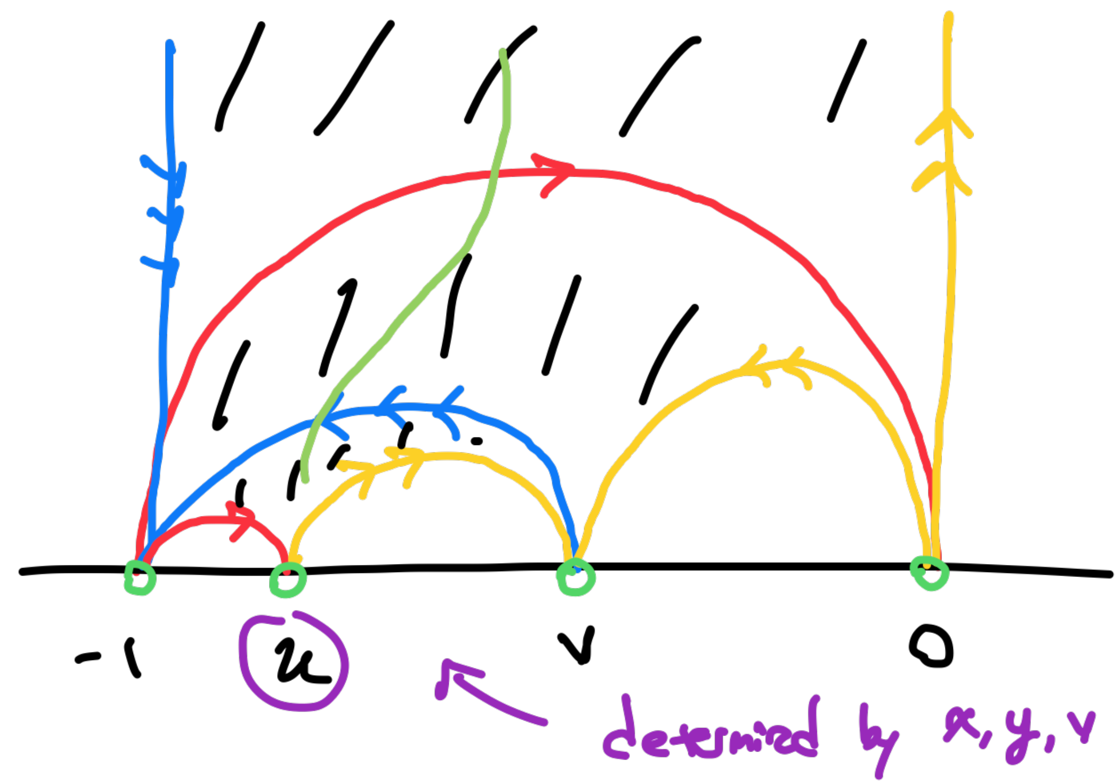
Another Example

M = 

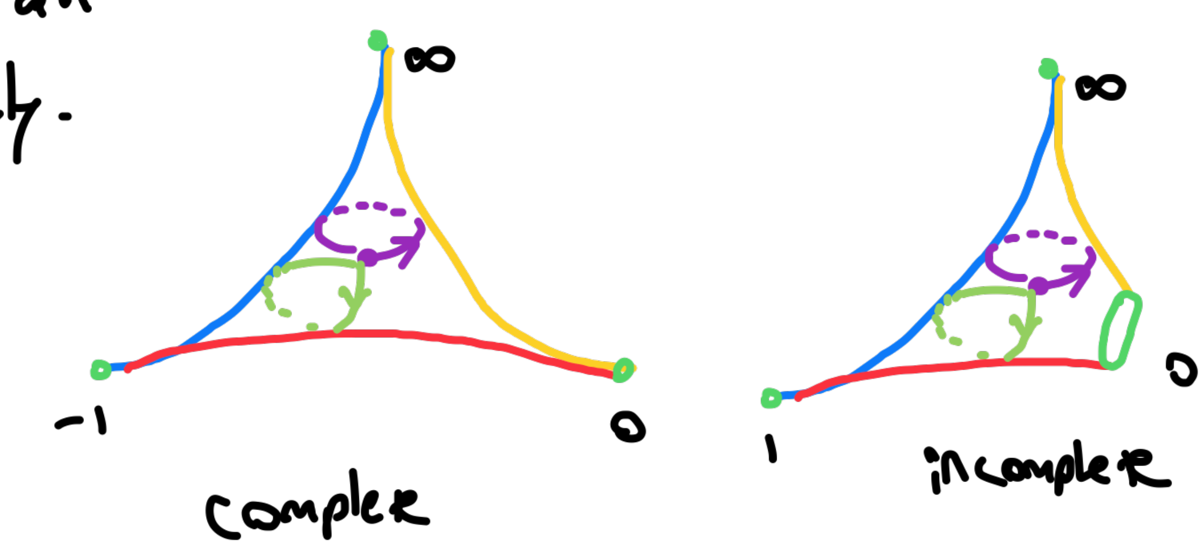
$\setminus \{p, q, r\}$, Topologically M = 



and

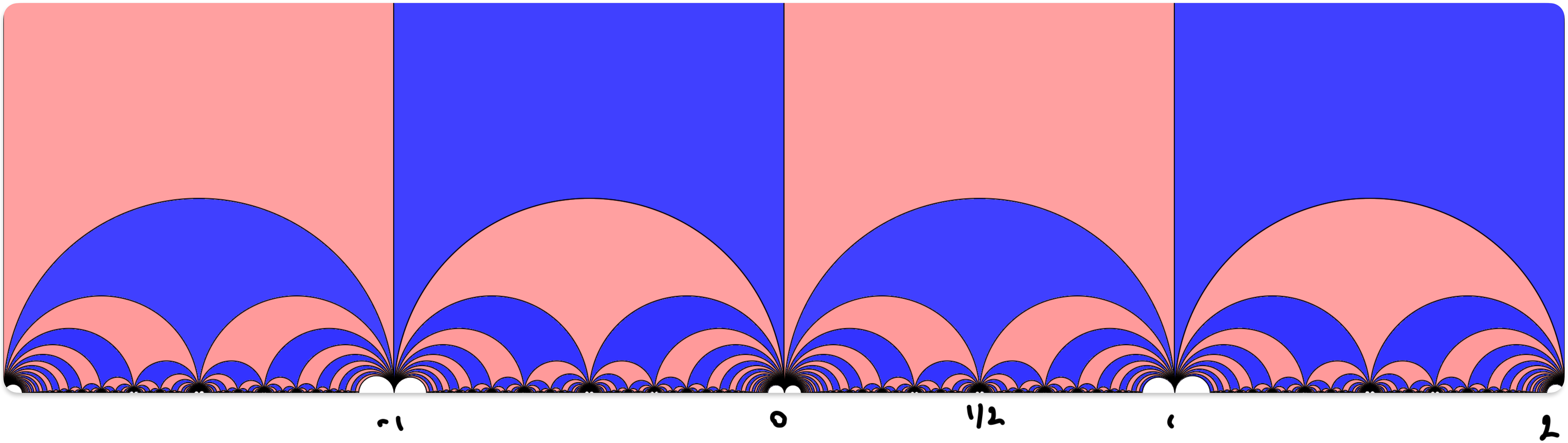


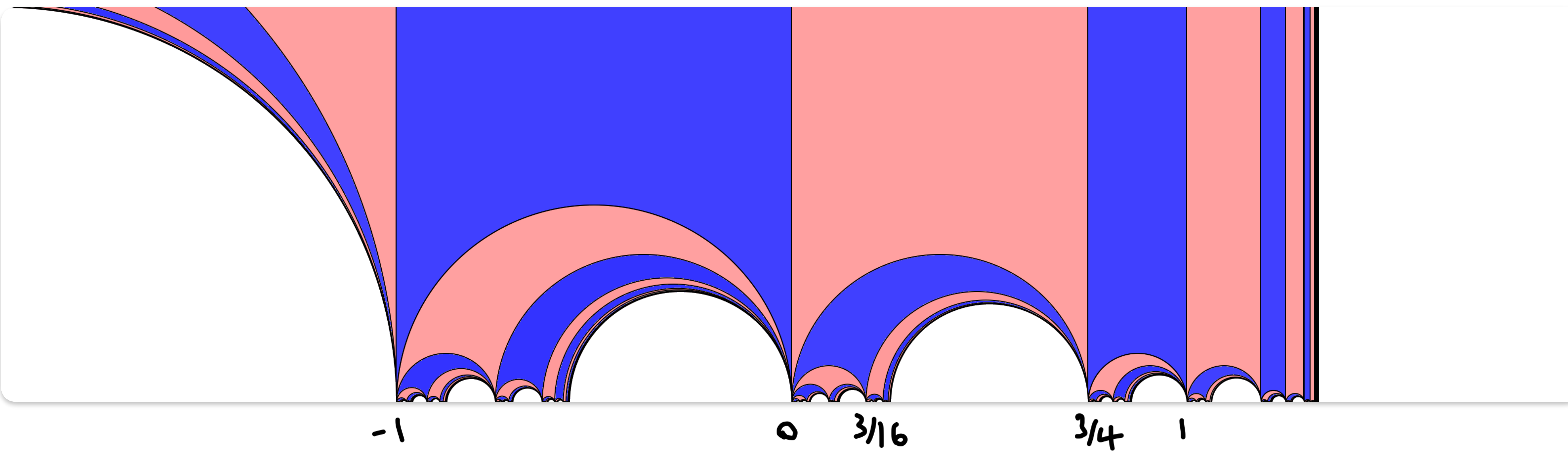
Complete structure if all geodesics extend indefinitely.



$\mathbb{H}^2(M)$ locally 3 dimensional.

Some Hyperbolic Structures on $S^2 \setminus \{p, q, r\}$





Ehresmann - Weil - Thurston

To each (G, X)-structure we assign on M up to equivalence, we obtain a representation of the fundamental group, $\pi_1(M, p) \xrightarrow{hol} G$ called holonomy map. Well defined up to conjugation

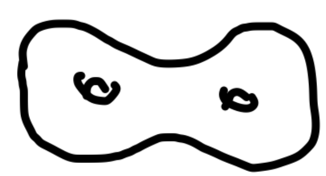
get a map

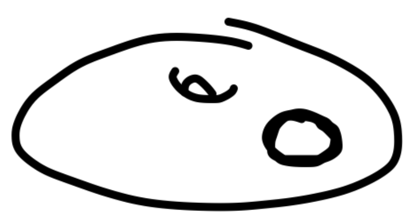
$$\frac{\{\text{Marked } (G, X)\text{- structures on } M\}}{\text{Equivalence}} \xrightarrow{Hol} \frac{Hom(\pi_1(M, p), G)}{G}$$

Ehresmann - Weil - Thurston

$\mathrm{Hom}(\pi_1(M,p), G)/G := \mathcal{X}(\pi_1(M,p), G)$ called <u>character</u> <u>variety</u>. Many tools used to study it and its topology.

A lot is known if...

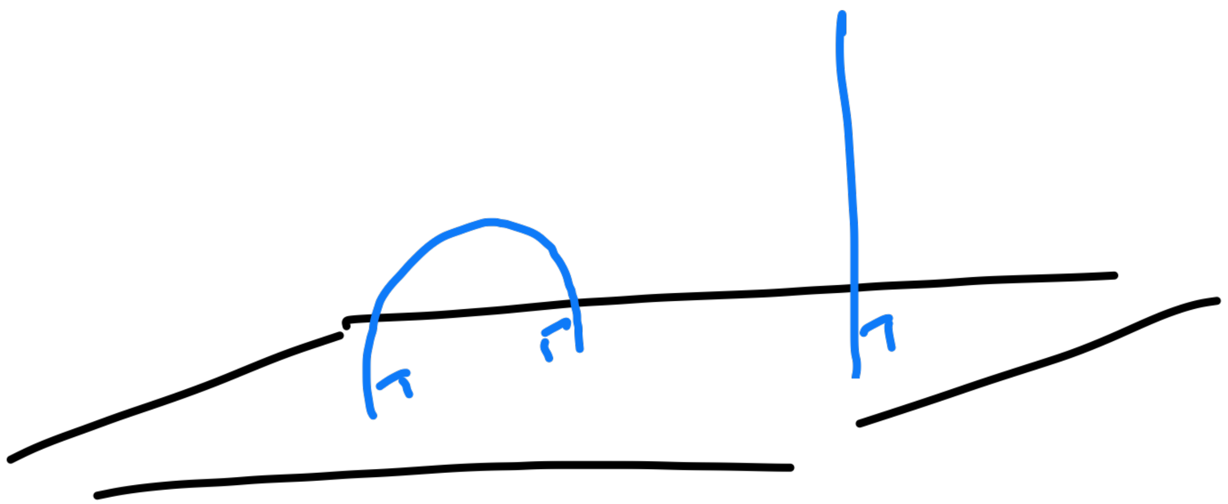
$M =$ surface  and $G = PSL(2,\mathbb{R})$

 with or without ∂M

For example in this case $\mathcal{X}(\pi_1(M,p), G)$ has finitely many components.

Hyperbolic 3-manifolds

Geometric manifolds with $G = PSL(2, \mathbb{C})$ and $X = \mathbb{H}^3$



$\mathbb{H}^3$, where $\partial \mathbb{H}^3 \cong \mathbb{C}P^1$

Can also think of as

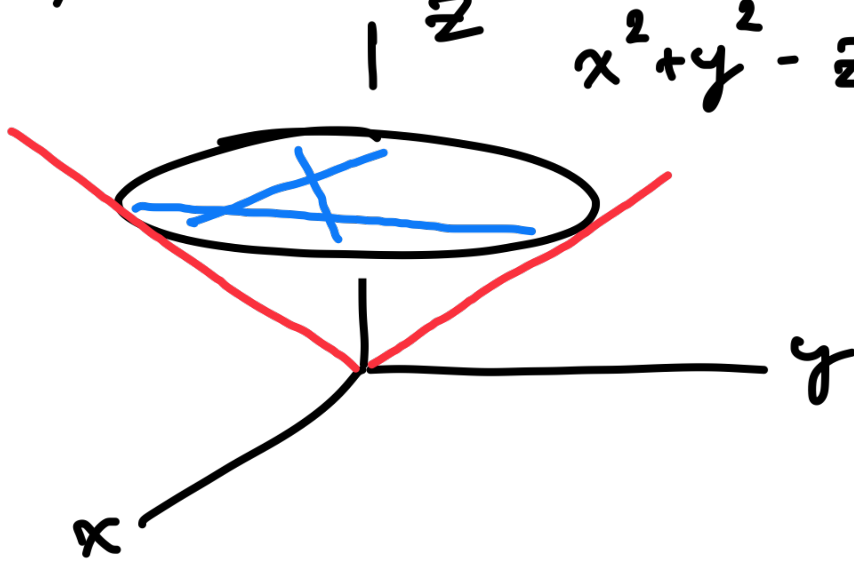
$$\mathrm{Proj}\left(\vec{x} \in \mathbb{R}^{3,1} \mid \langle \vec{x}, \vec{x} \rangle < 0 \right) = K^3$$

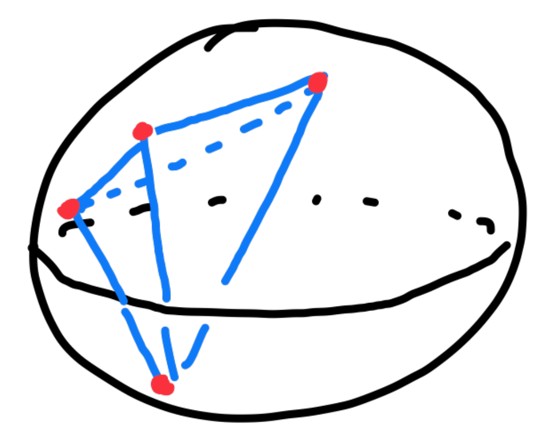
with $G = PSO(3,1)$

K^2 for illustration

Projectivize

$x^2 + y^2 - z^2 < 0$

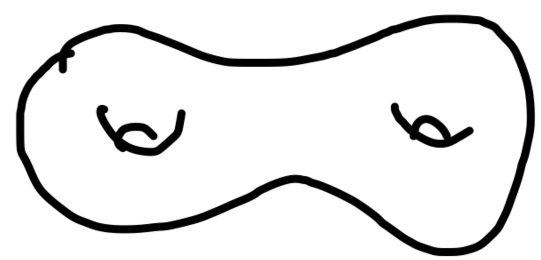


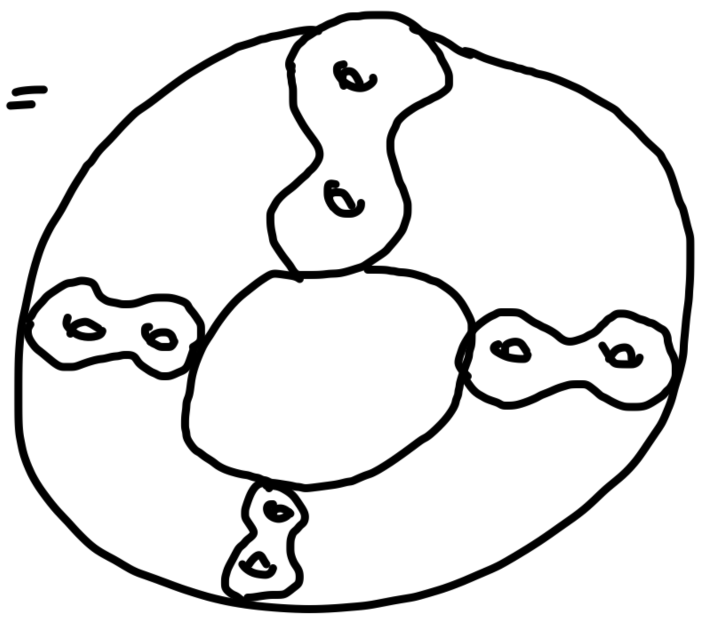


K^3 in affine chart of $\mathbb{R}P^3$

Where find these?

$M = S^3 \setminus$  famously has a hyperbolic structure.

More generally, $S =$ , $\phi =$ some pseudo Anosov

then form $M_\phi =$  mapping torus of ϕ.

By Thurston's Hyperbolization, M_ϕ admits a hyperbolic structure.

Question:

How many such hyperbolic structures on M_ϕ are there?

Mostow Rigidity: let M be an orientable 3-manifold. Any two complete finite volume hyperbolic structures on M are in fact isometric.

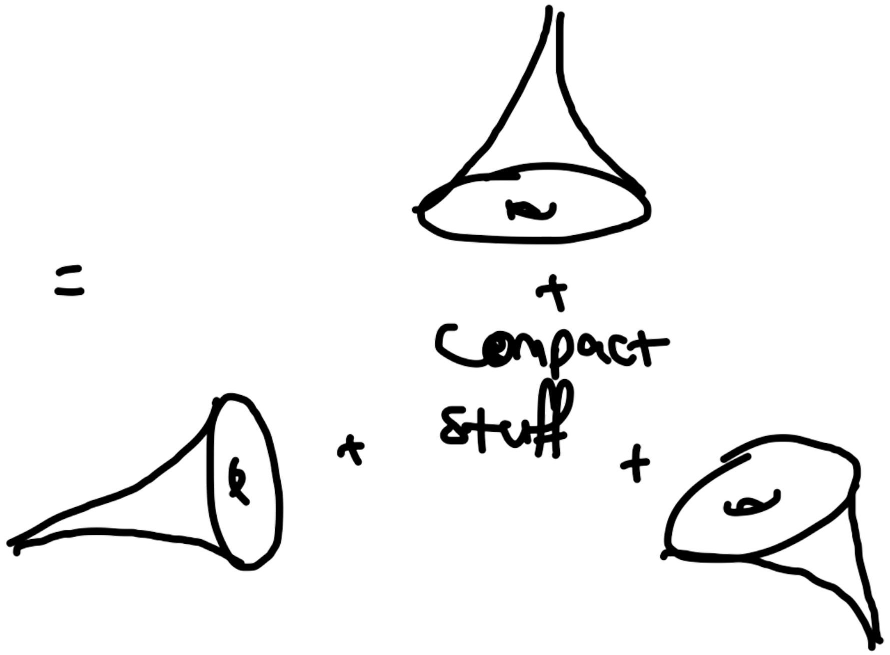
EWT: $\dfrac{\mathbb{H}^3_{\text{complete}}(M)}{\text{Equivalence}} \xrightarrow{\text{Hol}} \mathcal{X}_{\text{comp}}(\pi_1(M,p), PSL(2,\mathbb{C}))$

at complete structure is a point.

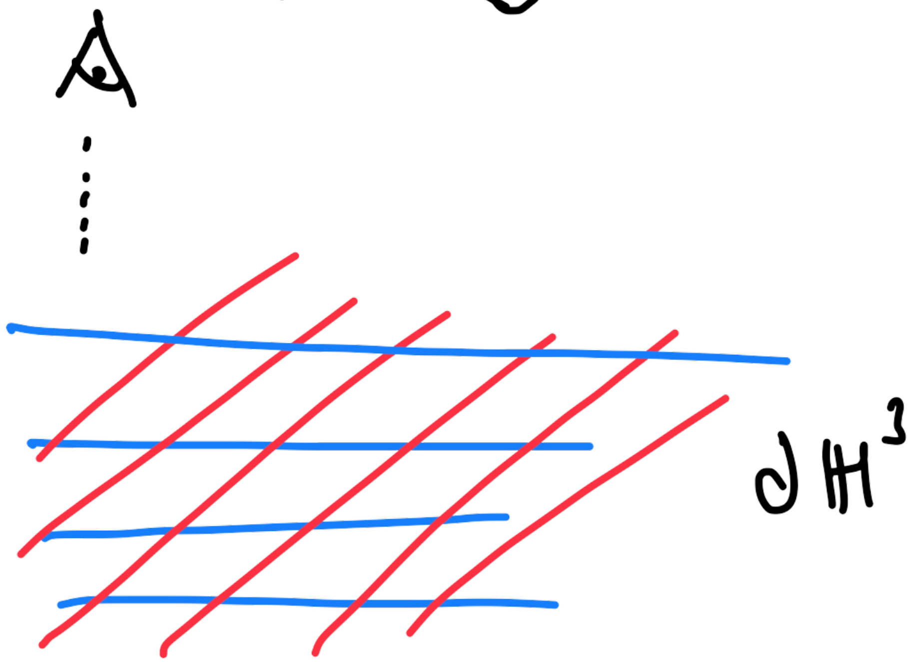
What about incomplete?

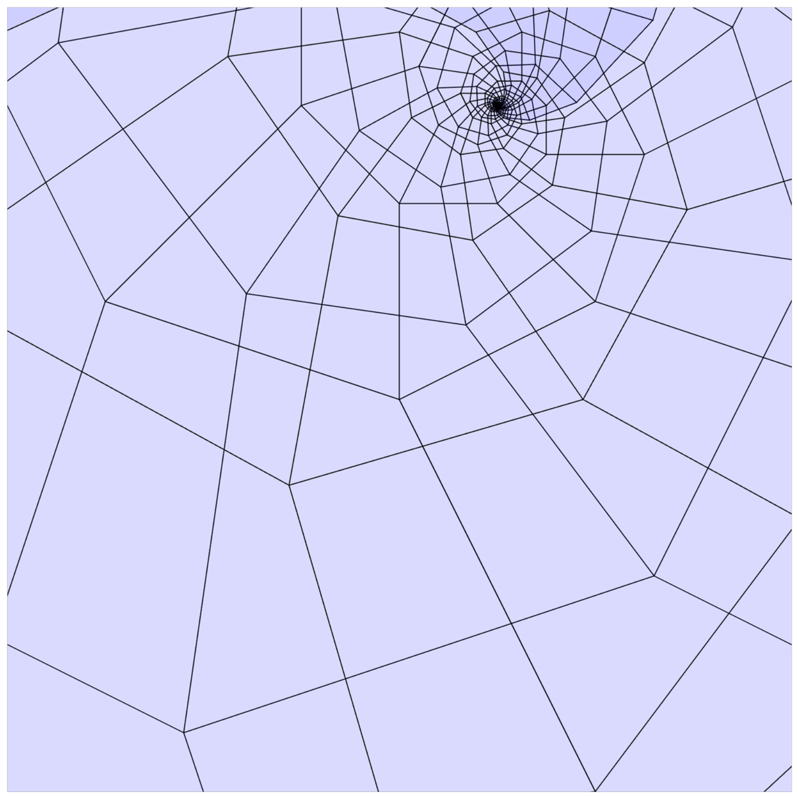
Thurston's Dehn-Filling says deformation space locally $\mathbb{C}^{\#\text{ of cusps}}$-dimensional.

if $M =$



then $\mathbb{H}^3(M)$ locally $\cong \mathbb{C}^3$ around complete hyperbolic structure.





What about Projective Deformations?

Every hyperbolic structure is naturally a projective one.

$$(\mathbb{K}^3, PSO(3,1)) < (\mathbb{RP}^3, PGL(4,\mathbb{R}))$$

Fundamental Question:

Does M admit projective deformations of its hyperbolic structure? If not we say M is projectively rigid.

State of the Art

If M is compact, the answer is typically no
(unless M has a totally geodesic embedded surface)
CL(Th): Census manifolds admit at most 61 of 4,500 or so.

If M cusped, the answer is maybe
CL(Ti): So long as the cusps can deform and preserve convex subset of $\mathbb{RP}^3$, then yes, however explicit calculations are difficult.

State of the Art (cont)

There's a cohomological condition known as inf. proj. rigidity. rel cusp, (HP),

$$H^1(M; \mathfrak{sl}_4\mathbb{R}_{Ad \circ \rho}) \xrightarrow{res} H^1(\partial M; \mathfrak{sl}_4\mathbb{R}_{Ad \circ \rho})$$

Under this condition, M guaranteed to admit projective deformations and ∂M admits a (very nice type of) projective structure (BDL).

And if M is (iprrc) infinitely many Dehn-Surgeries on M are projectively rigid (HP).

Some My Work

2000 or so Dehn-Surgeries of fig. eight-knot complement are proj. rigid.

$F_2 \hookrightarrow \Gamma \twoheadrightarrow F_1$

iprrc of a hyp. once punctured torus bundle can be equivalently stated in terms of dynamics on $\mathcal{X}(\pi_1(F), PGL(4,\mathbb{R}))$ at point corresponding to complete hyp. structure.

iprrc of a hyp. once punctured torus bundle can be determined from roots of a polynomial associated to holonomy of $\pi_1(M_\phi) \hookrightarrow PSL(2,\mathbb{C}) \cong PSO(3,1) < PGL(4,\mathbb{R})$

Ideas and Techniques

Study deformation space at infinitesmal level $T_{[\rho]}\mathcal{X}(\Gamma, PGL(4,\mathbb{R})) \cong H^1(\Gamma; \mathfrak{sl}_4\mathbb{R})$
(Weil inf. rig.)

Try understand deformations of boundary torus,
investigate $H^1(M; \mathfrak{sl}_4\mathbb{R}) \xrightarrow{res} H^1(\partial M; \mathfrak{sl}_4\mathbb{R})$
image is 3-dimensional due ½-lives ½ dies

Numerically investigate using SnapPy, interval arithmetic, etc.
Plot using computers.

Thank you for your time!