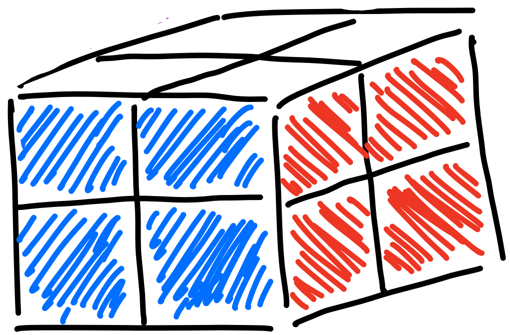
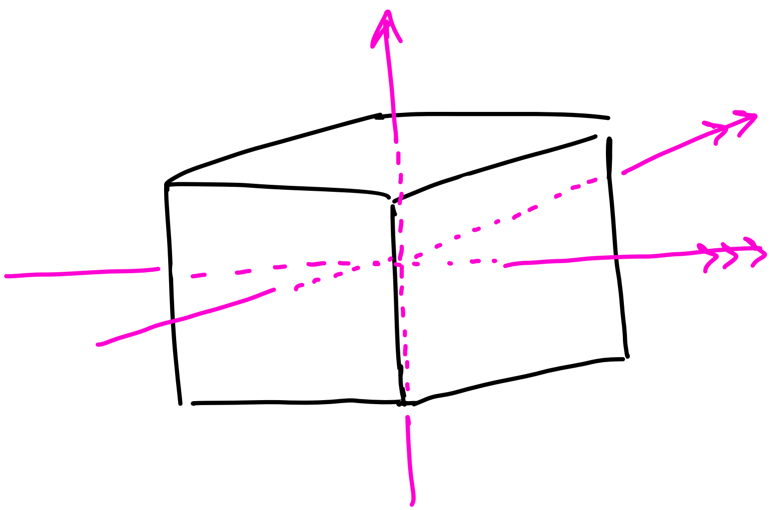
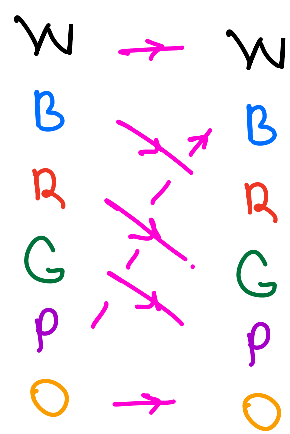
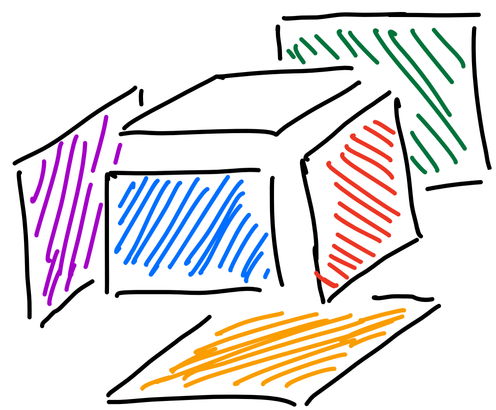
Some Group Theory and the 's Cube

Feb 19th, 2025 - Providence College







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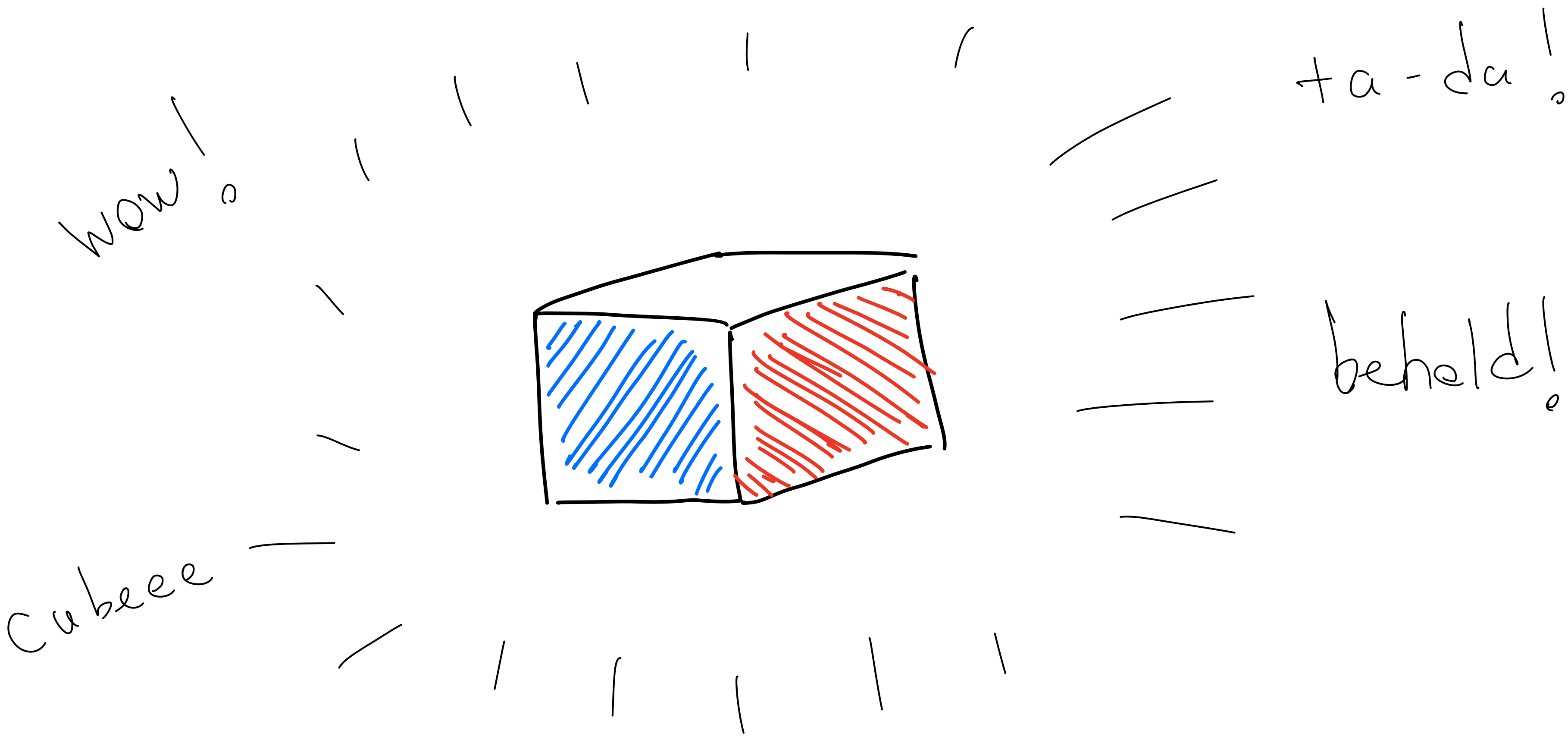
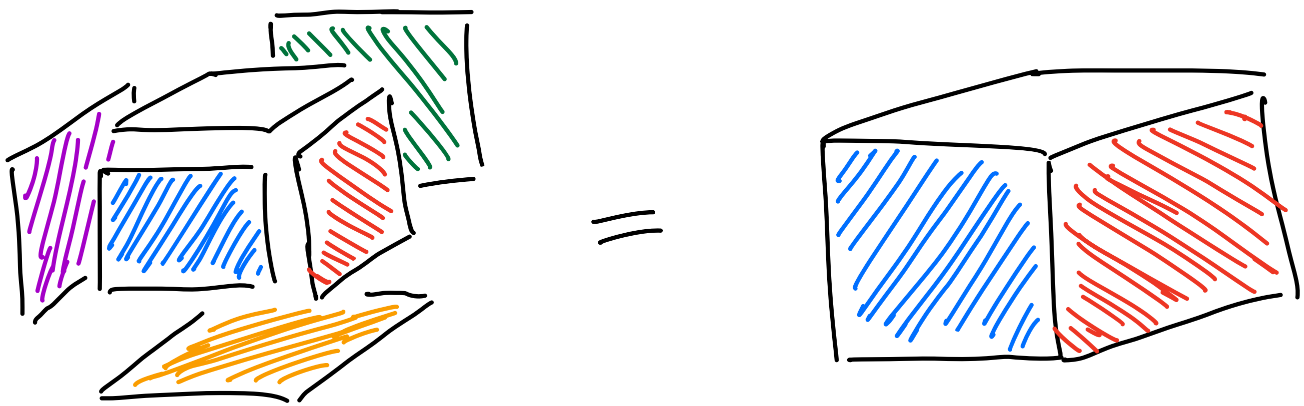
Overview of Talk

- Detailed 1st example
- Special group #1
- Special group #2
- Combine work for results

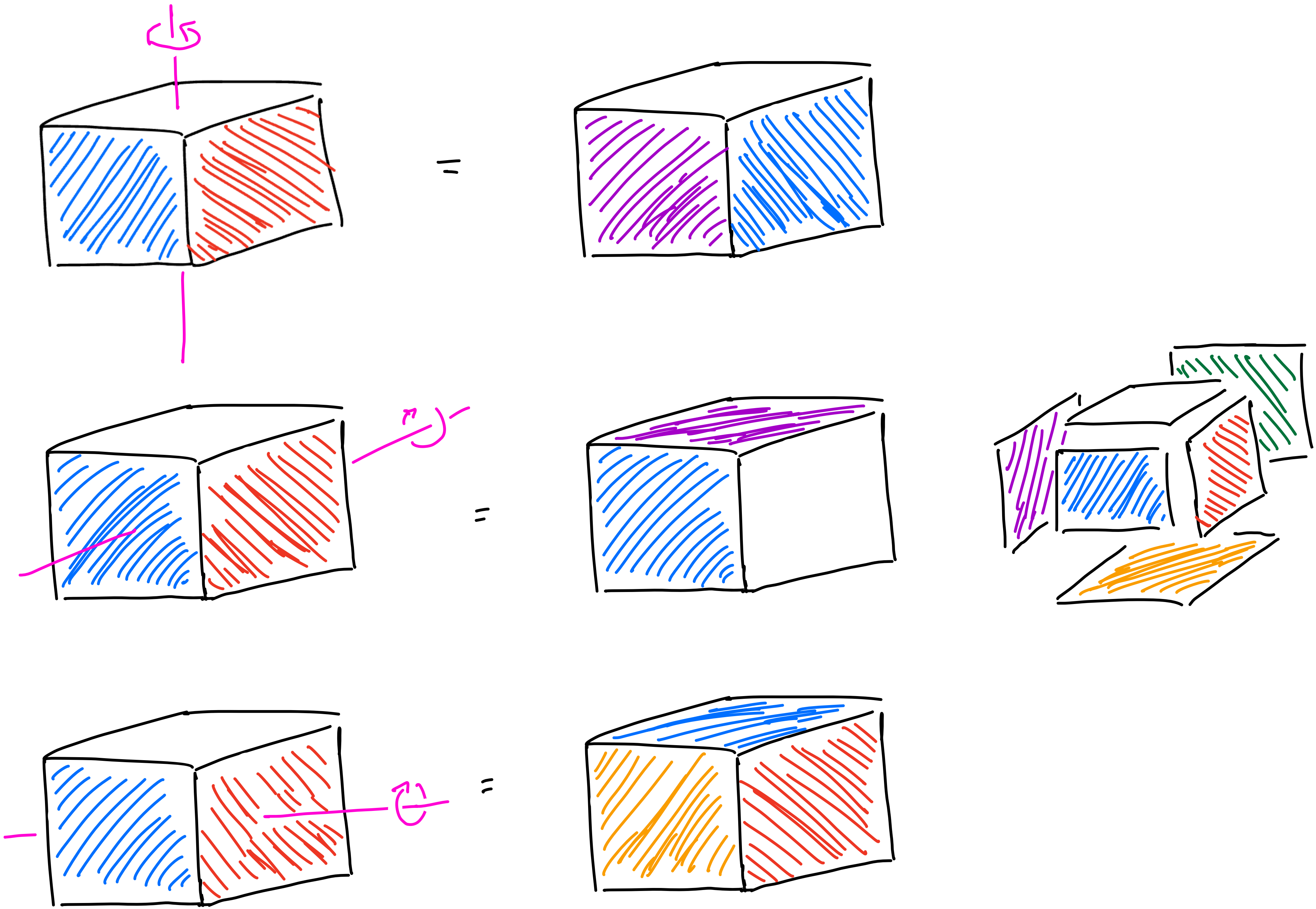
Before trying to solve the $2 \times 2 \times 2$ try to solve the $1 \times 1 \times 1$!



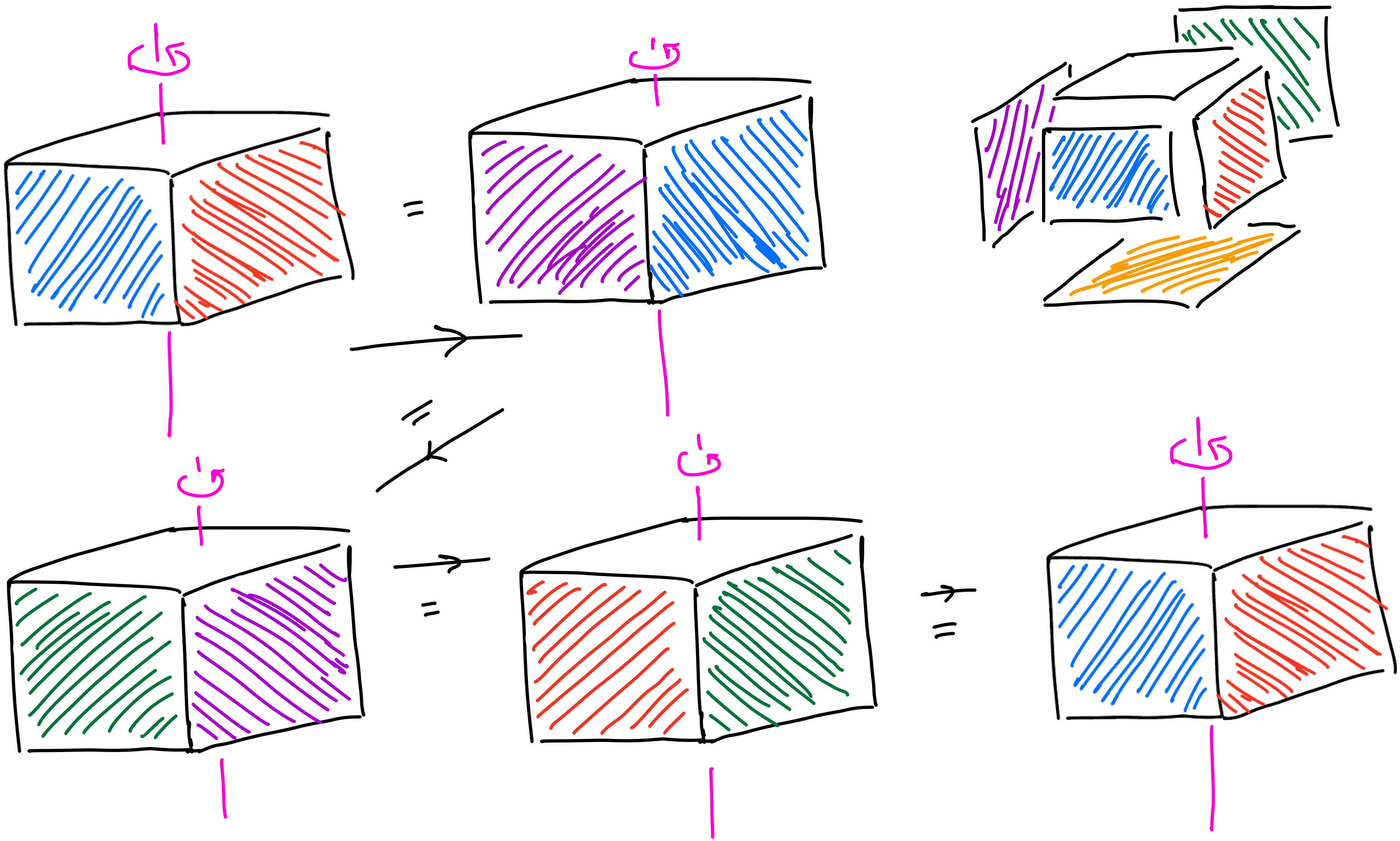
Question: How many symmetries of cube are there?



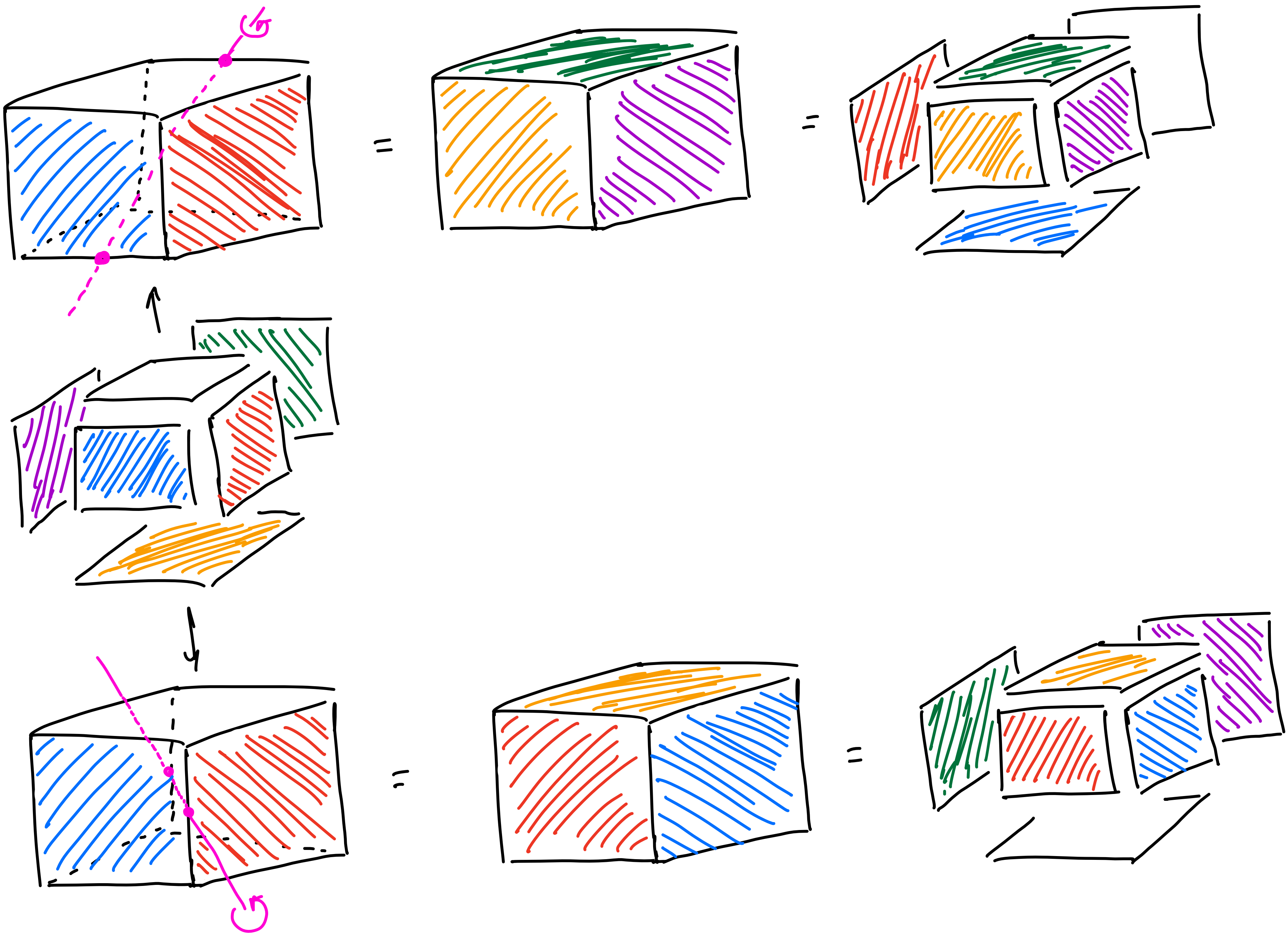
First type of rotation :



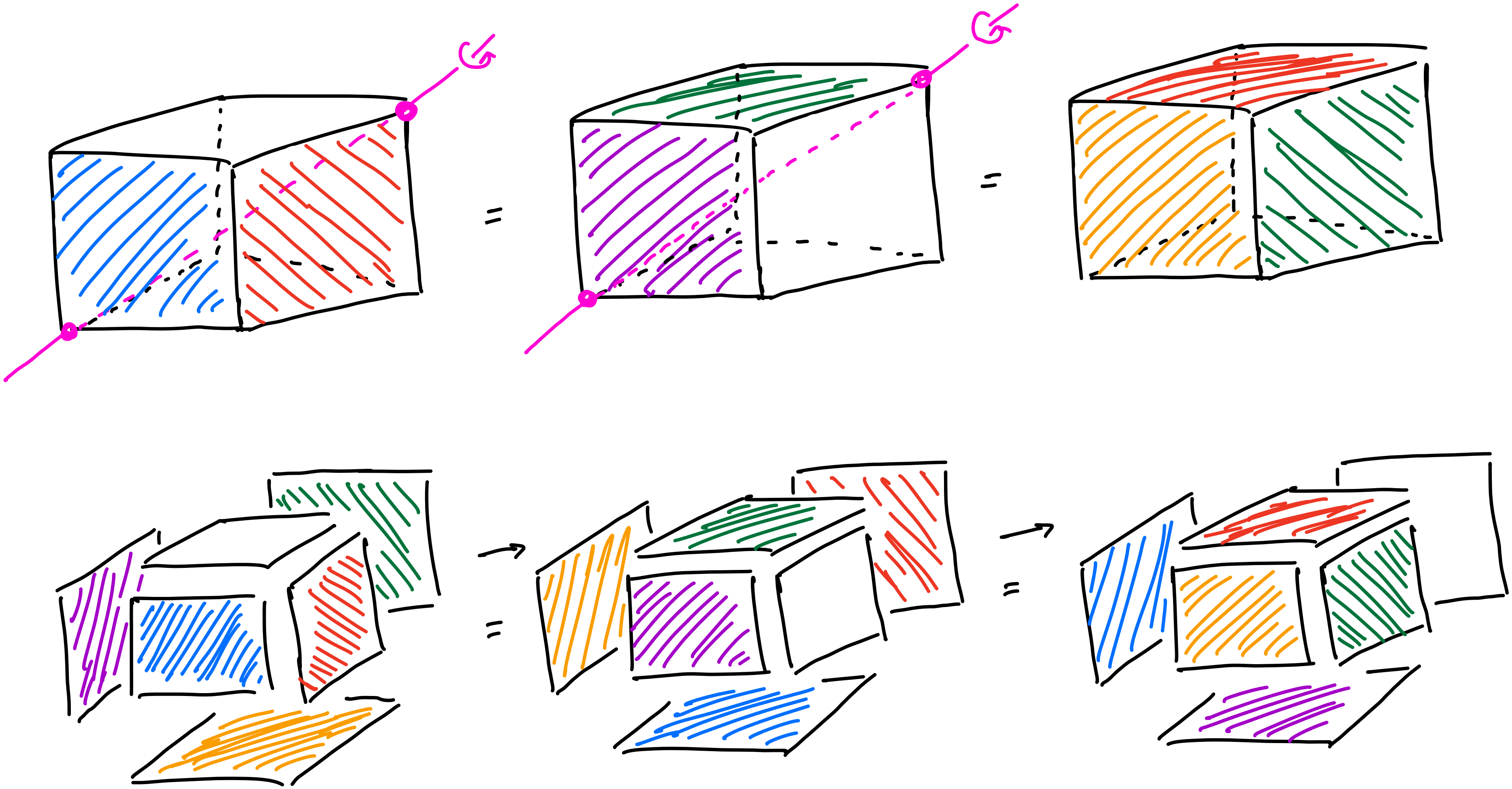
If you do it 4-times ...



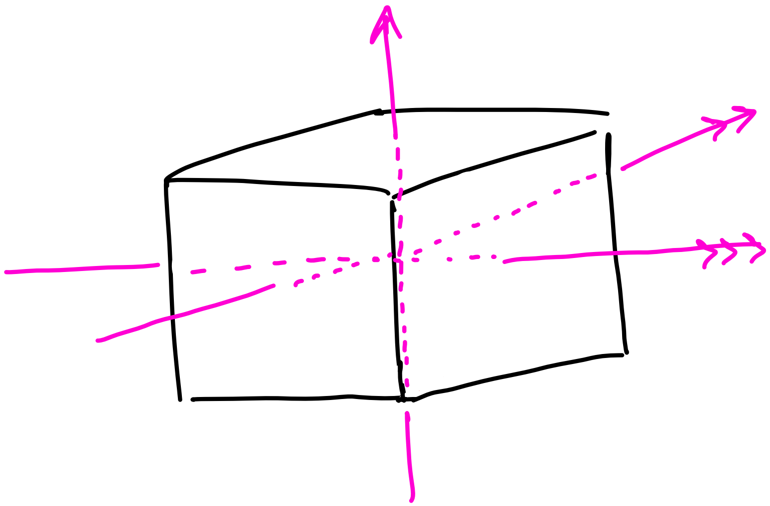
Second type of rotation



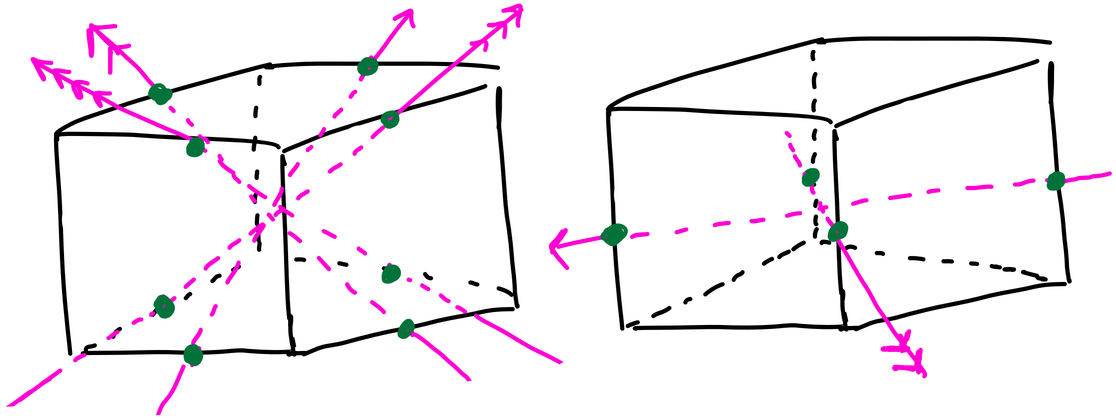
Third type of rotation :



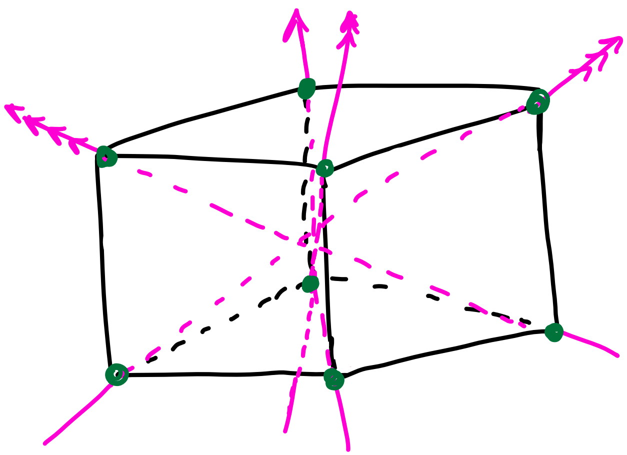
1st type: 3 axes of symmetry



2nd type: 6 axes of symmetry



3rd type: 4 axes of symmetry



1st type: (3 axes)(3 non-trivial) = 9

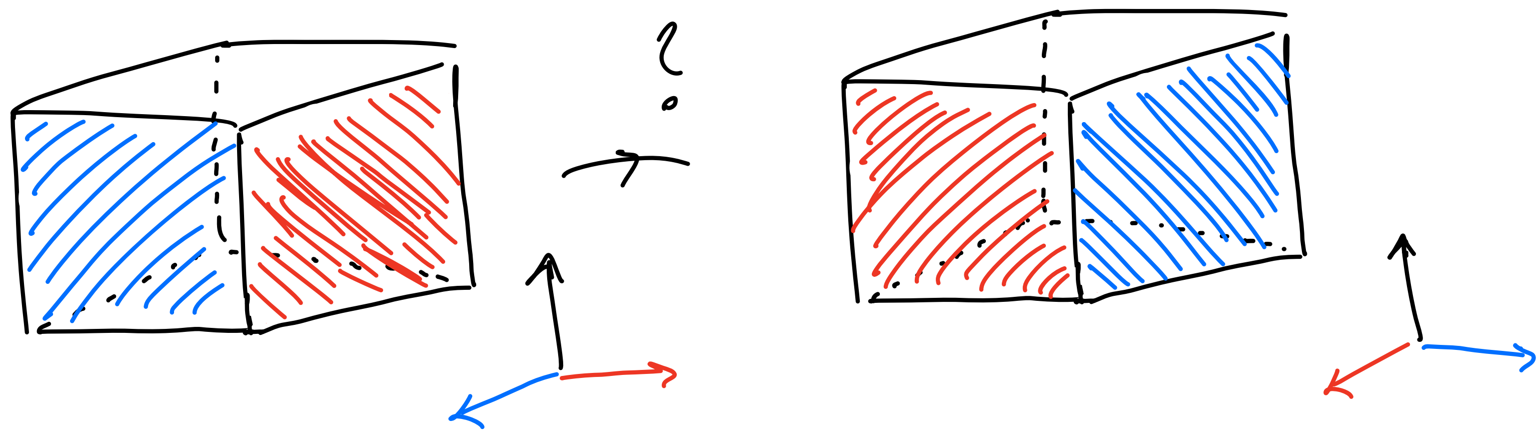
2nd type: (6 axes)(1 non-trivial) = 6

3rd type: (4 axes)(2 non-trivial) = 8

4th type: (nothing) = 1

24 distinct symmetries

Is that all? What about ?



Let G = all possible symmetries from rotating the cube.

Function: $G \longrightarrow$ Faces of Cube

$f(g)$ = Color g sends blue face to.

Observation: Certainly $G \twoheadrightarrow$ Faces of cube.

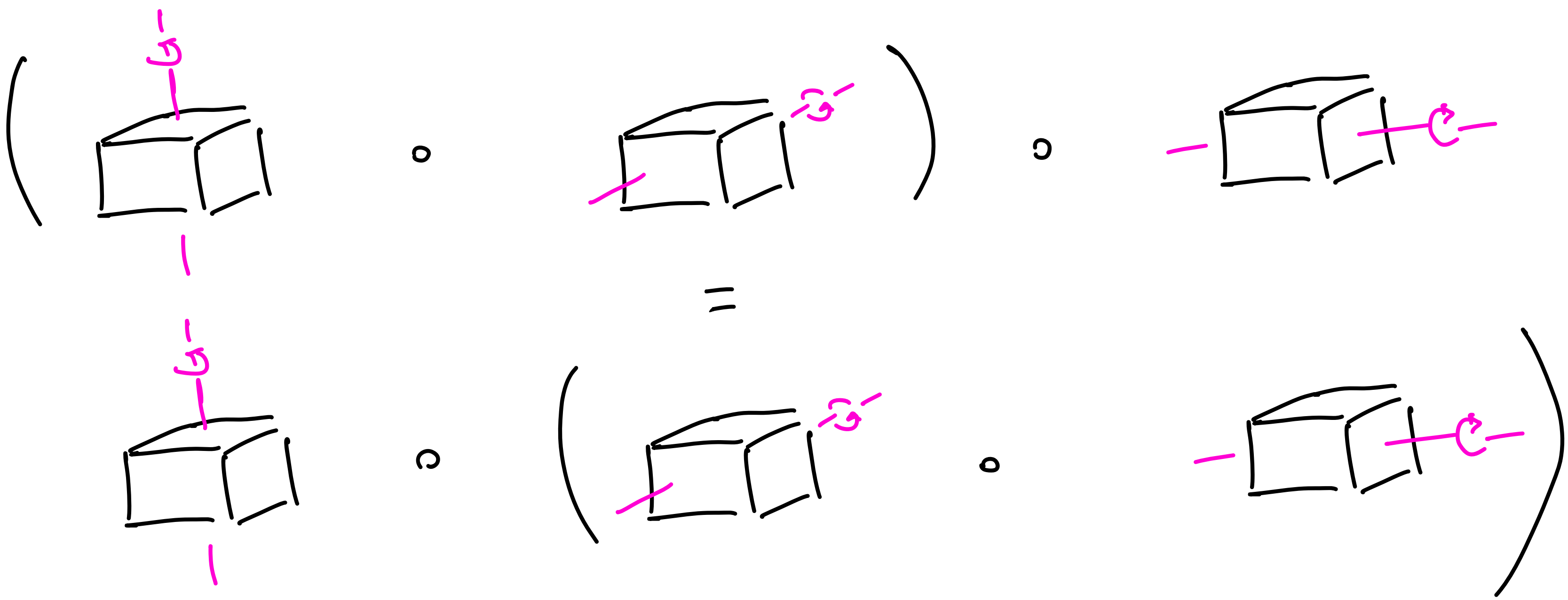
$|G|$ = # g sends blue to blue + # g sends blue to red

+ ... + # g sends blue to purple.

$= 4 + .. + 4 = 4 \cdot 6 = 24$ symmetries //

Observation: the symmetries of the cube have algebraic structure.

1. Given two symmetries you can combine them to get another
2. Given any symmetry you can always undo it
3. There's a symmetry that does nothing
4. Given any ordered list of symmetries, it doesn't matter how you choose to reduce it, e.g.



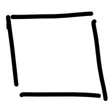
An object that satisfies these rules is called a group. Groups are everywhere!

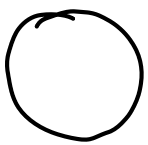
ex: $(\mathbb{Z}, +)$

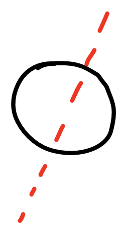
non-ex: $(\mathbb{Z}, -)$

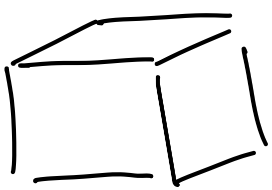
$(2-1)-1 = 0$

$2-(1-1) = 2$

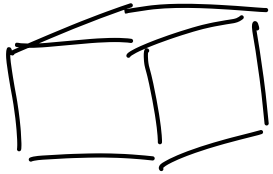
ex: (, multiple of 90° rotations)

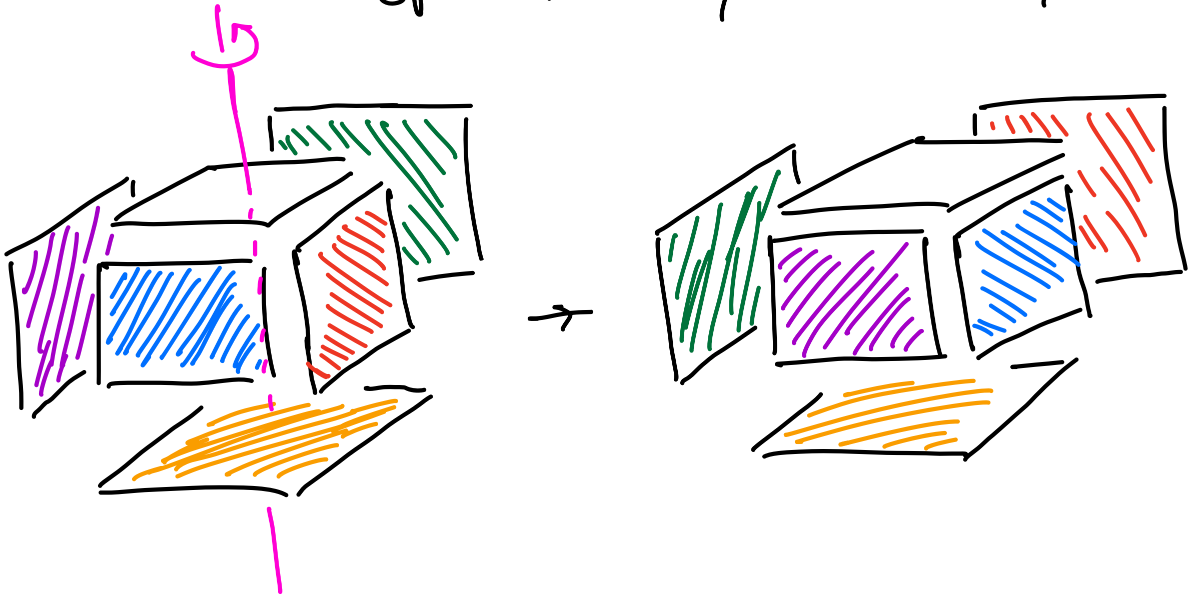
ex: (, any rotation)

ex: (, any rotation <u>or</u> reflection)

ex: (, rotations)

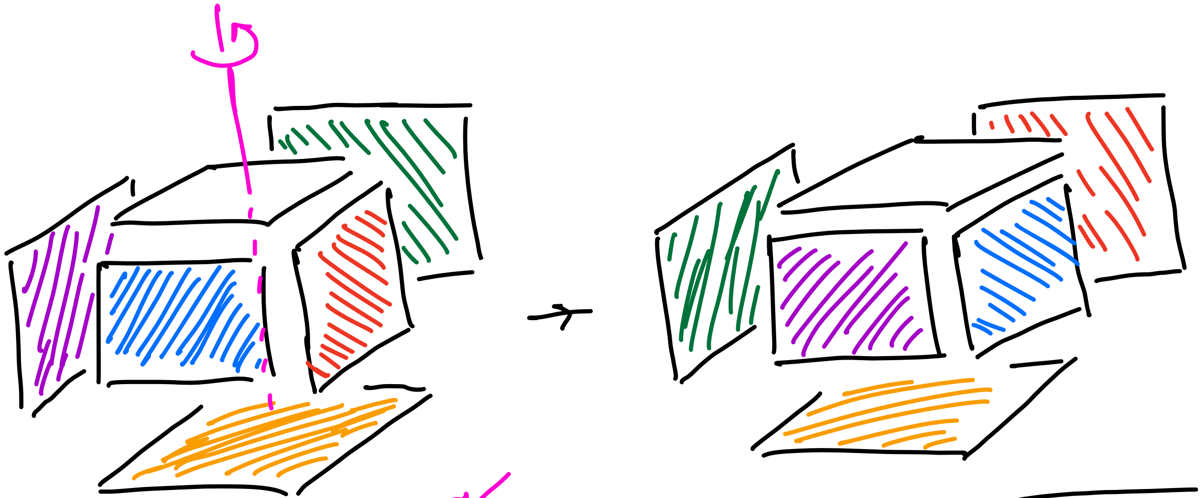
ex: (invertible linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$)

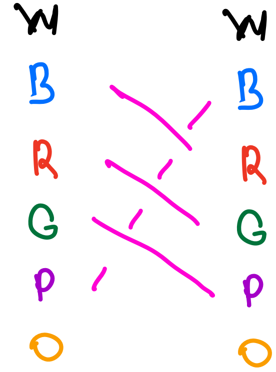
Returning to  we found 24 symmetries that could be specified by what they do to <u>faces</u>.



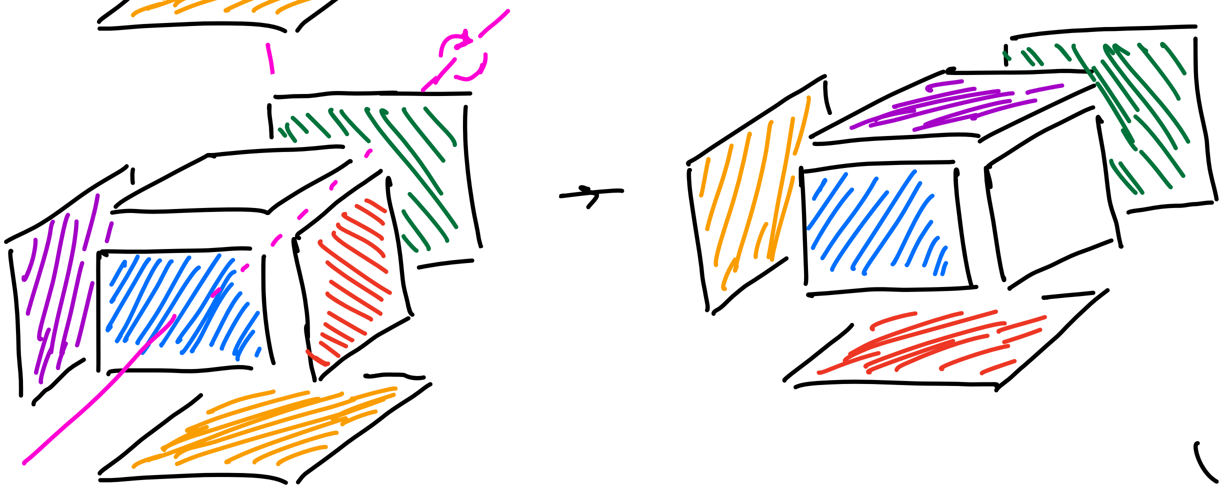
$$W \to W,\quad B \to R,\quad R \to G,\quad G \to P,\quad P \to B,\quad O \to O$$

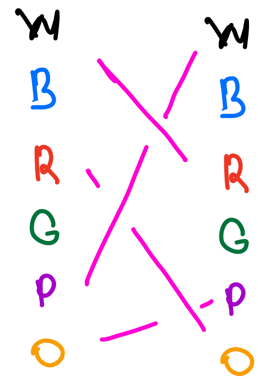
$$\sim \quad 1 \to 1,\quad 2 \to 3,\quad 3 \to 4,\quad 4 \to 5,\quad 5 \to 2,\quad 6 \to 6 \quad \sim \quad (2\,3\,4\,5)$$





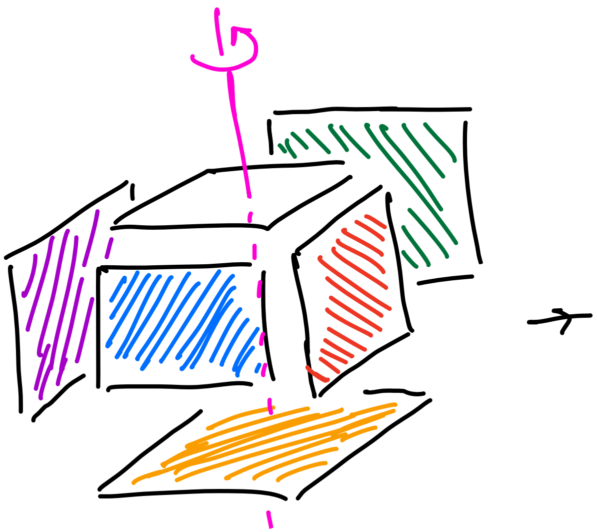
$\sim (2\,3\,4\,5)$

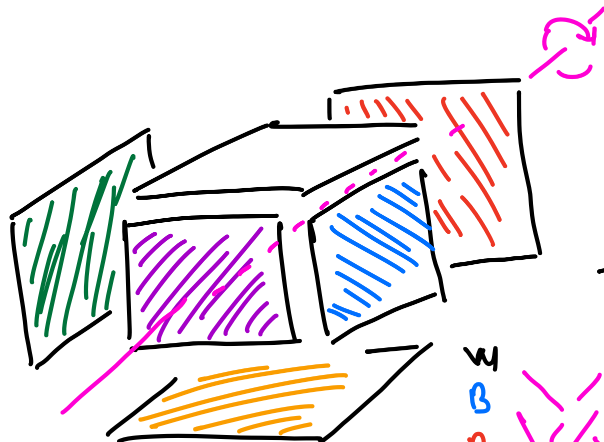


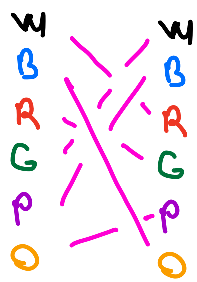


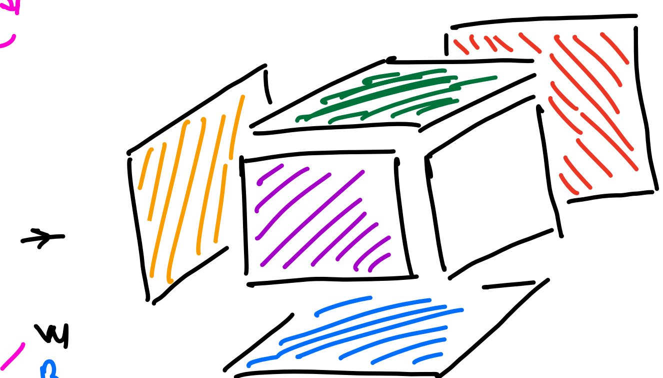
$\sim (1\,3\,6\,5)$

$(1\,3\,6\,5)(2\,3\,4\,5) = (2\,6\,5)(1\,3\,4)$



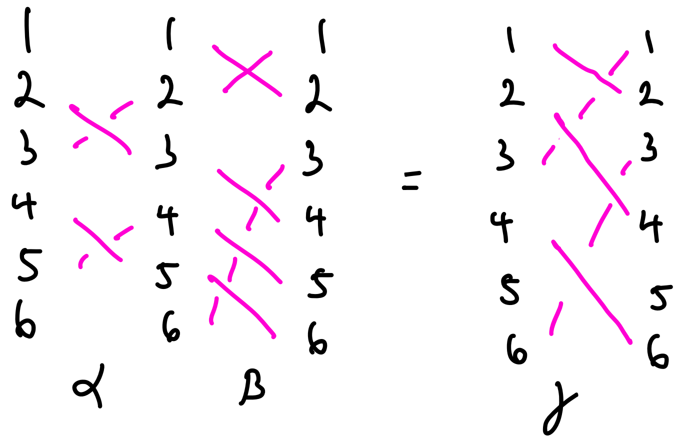






W → R   G → W
B → O   P → B
R → G   O → P

Here's another group: All rearrangements of $\{1,2,3,4,5,6\}$.



$\beta\alpha = \gamma$ (careful order)

$\alpha = (23)(45)$

$\beta = (12)(3456)$

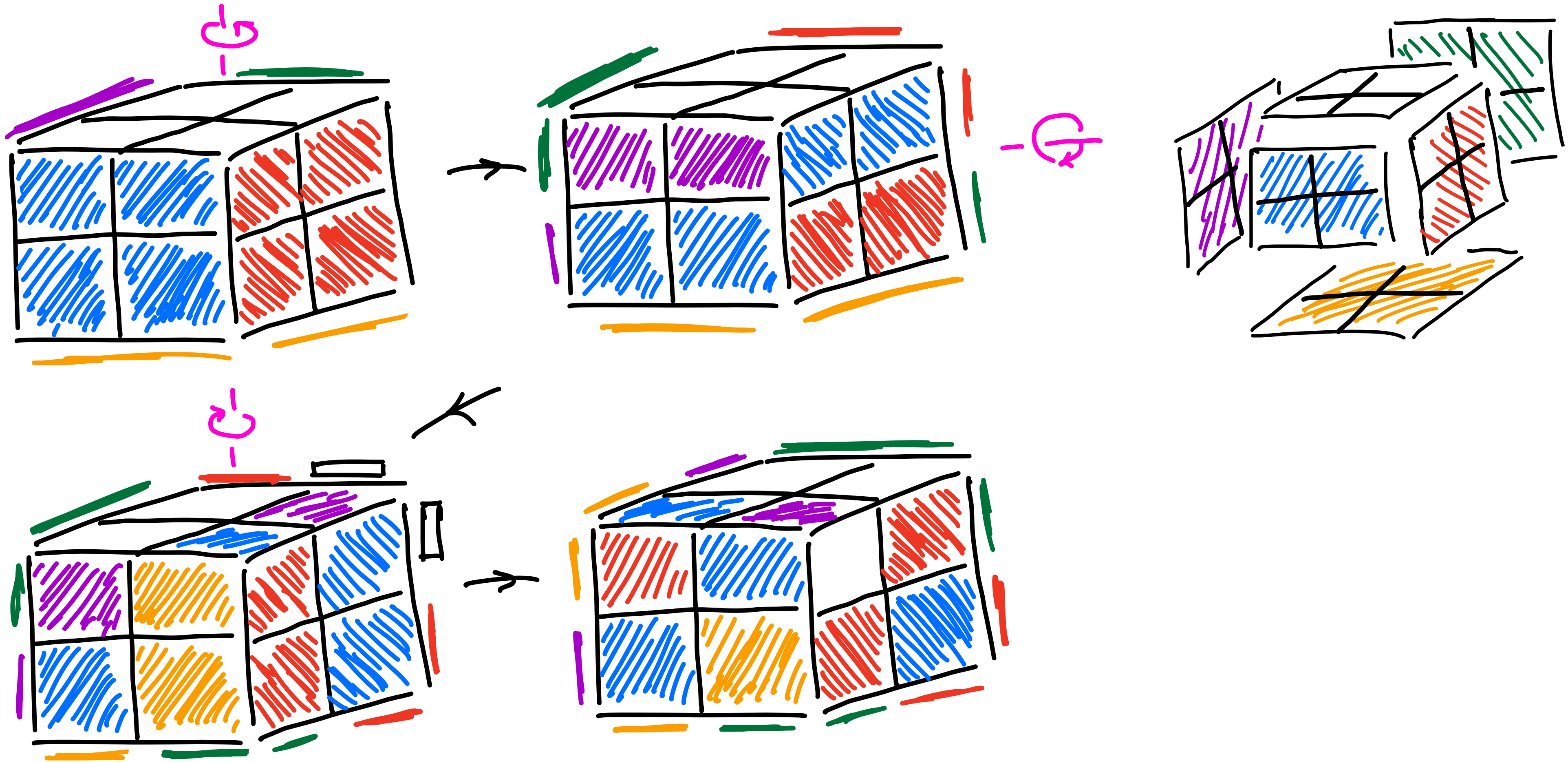
$\beta\alpha = (12)(3456)(23)(45)$

$= (12463) = \gamma$

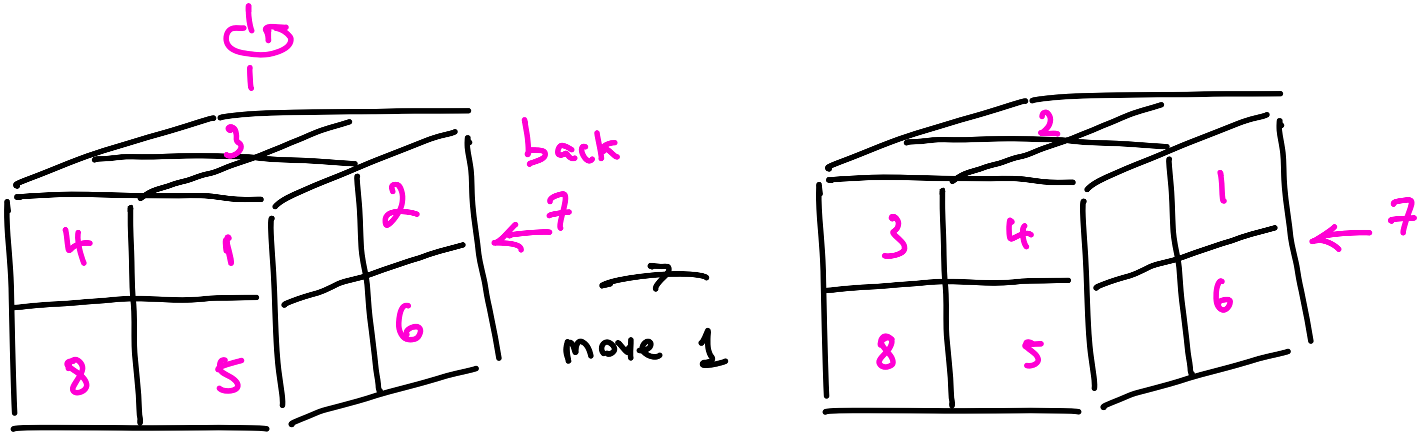
Denote this group by S_6. Readily generalize to S_n for any $n \geq 1$. Called the symmetric group on n-letters.

So we have a way to realize $G \hookrightarrow S_6$. We say that G embeds in S_6, or G is a subgroup of S_6.

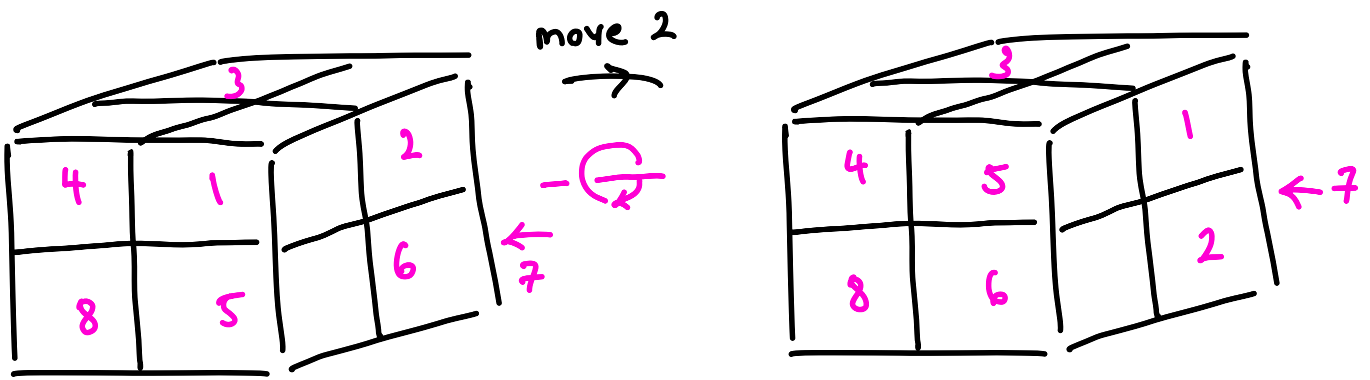
The 2x2x2 Group



Call this group G

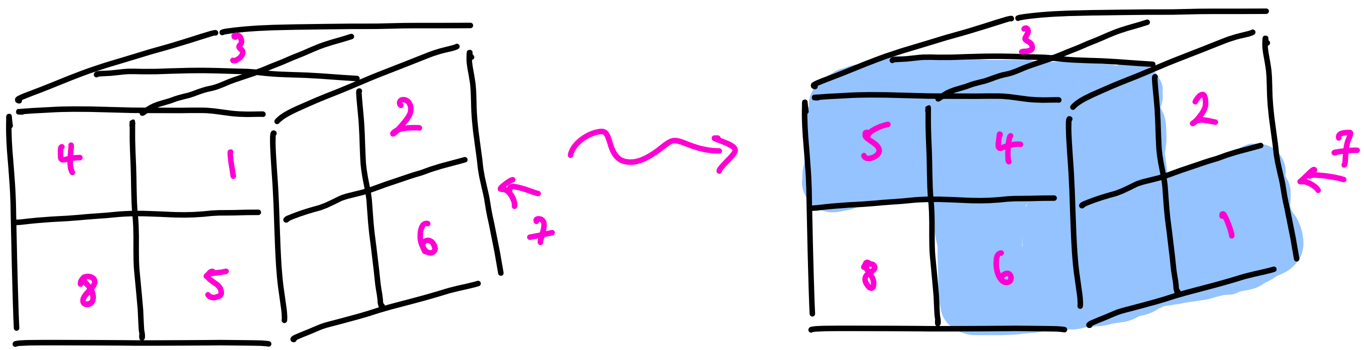


$= (1234)$



$= (1265)$

$$(4321)(1265)(1234) = (1654)$$



Quotient:

Have a function $G \xrightarrow{\phi} S_8$ that pretty much just forgets colors and only remembers cubes. Function has special property.

$$\phi(\underbrace{\text{move } 2 \circ \text{move } 1}_{\in G}) = \underbrace{\phi(\text{move } 2)}_{\in S_8} \circ \underbrace{\phi(\text{move } 1)}_{\in S_8}$$

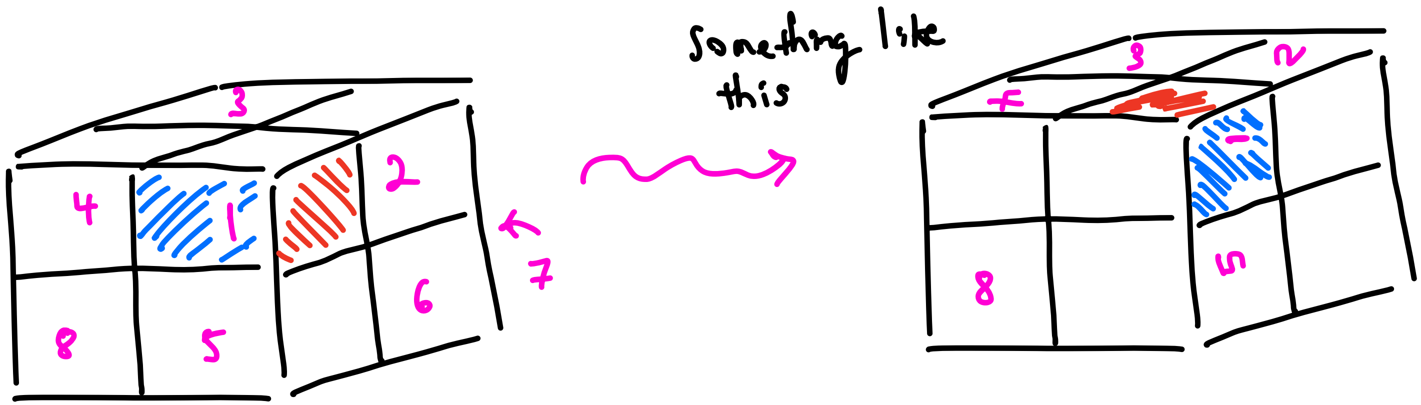
Function is called a homomorphism.

Fact: $\phi : G \twoheadrightarrow S_8$ is onto (surjective)

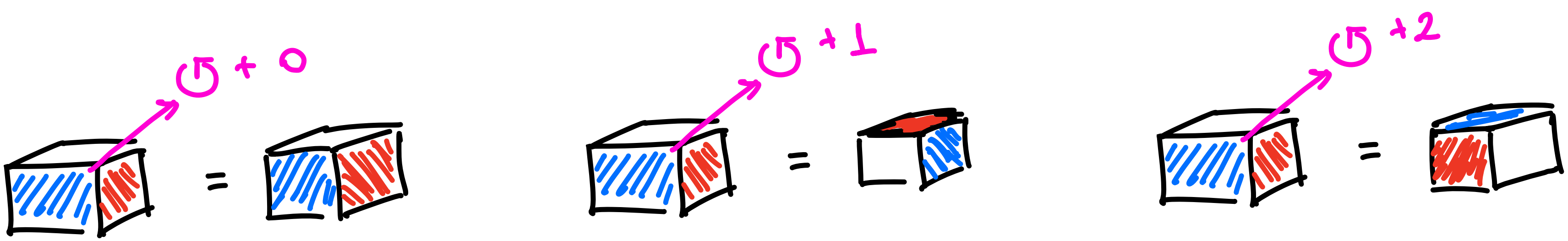
We call such homomorphisms quotients.

Special subgroup #1

Consider our $G \xrightarrow{\phi} S_8$. The subgroup $K \le G$ so that $\phi(k) = id_8$ has a nice description.



Could describe each such move by a list of 8 twists counterclockwise. Each twist is a rotation by 120° counterclockwise.

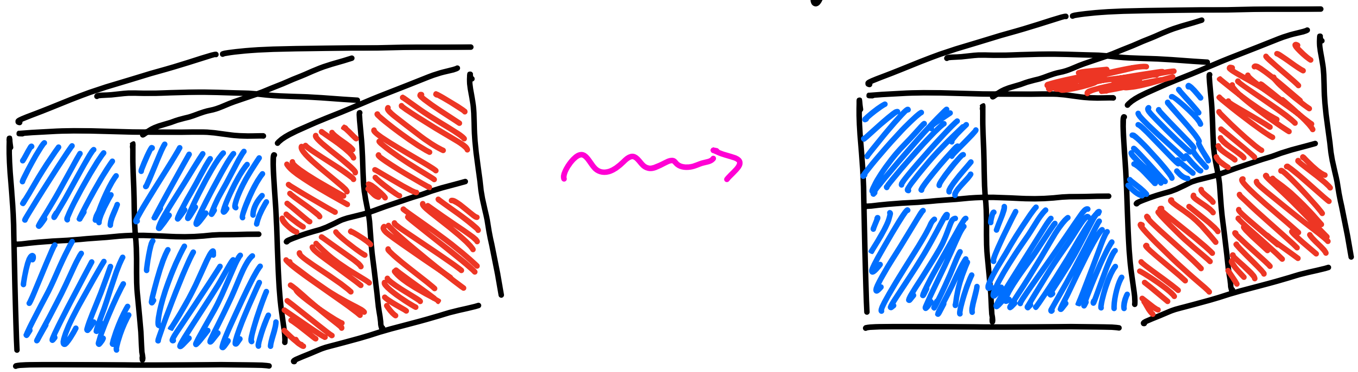


Each twist an element of $\mathbb{Z}_3 \cong \{0°, 120°, 240°\}$
$\cong \{0, 1, 2\}$

So have embedding $K \hookrightarrow \mathbb{Z}_3^8$. Is <u>every</u> twist combination possible?

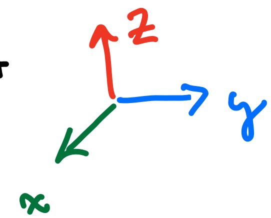
Example:

fix everything but one and twist $120°$ cc.

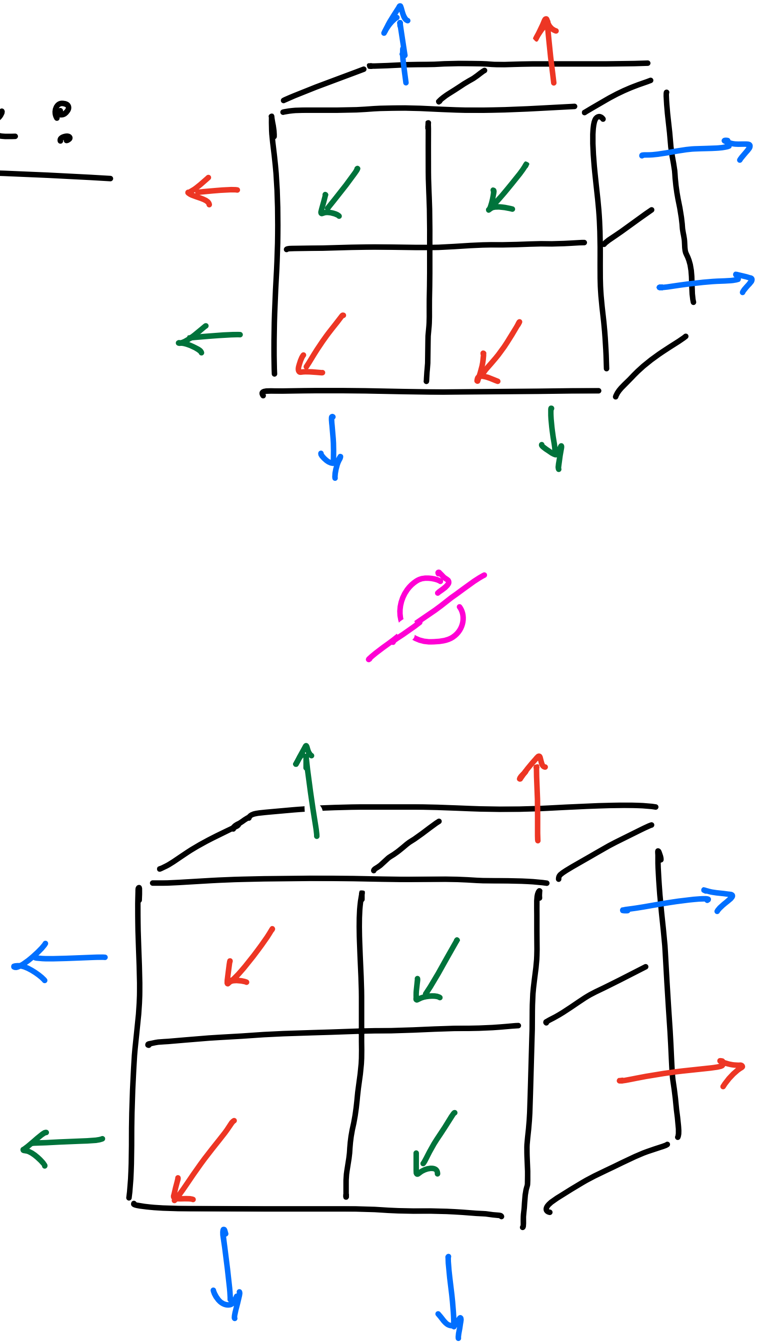


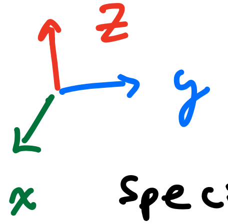
Local orientations:

For every cube, let's assign an 'up' or a local orientation. Pretty much just where z-pointing away from cube.

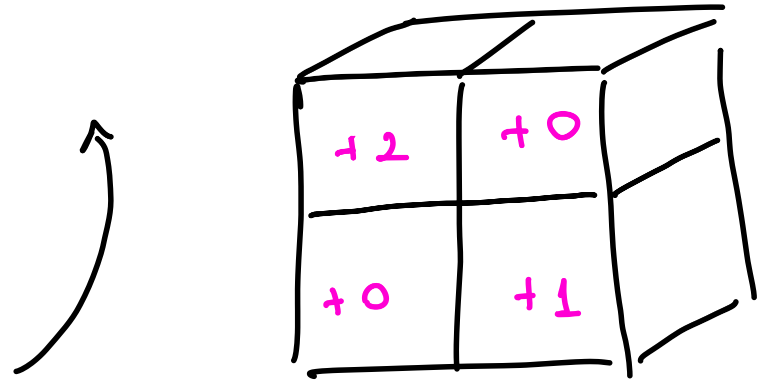


example :

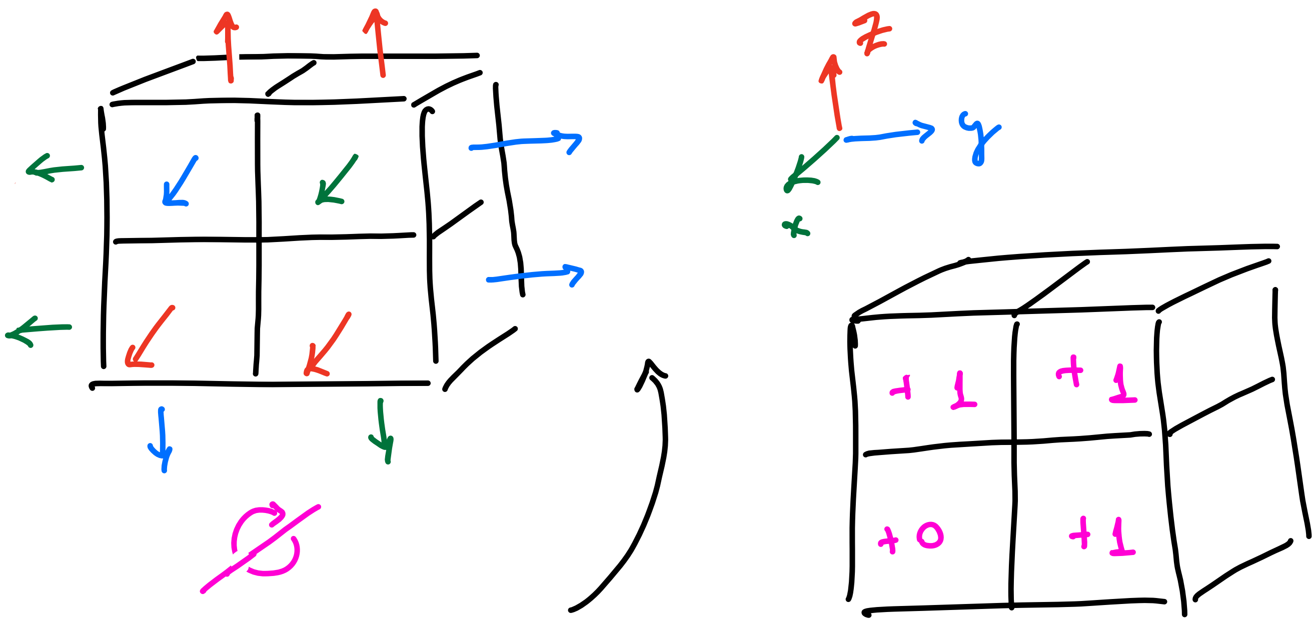


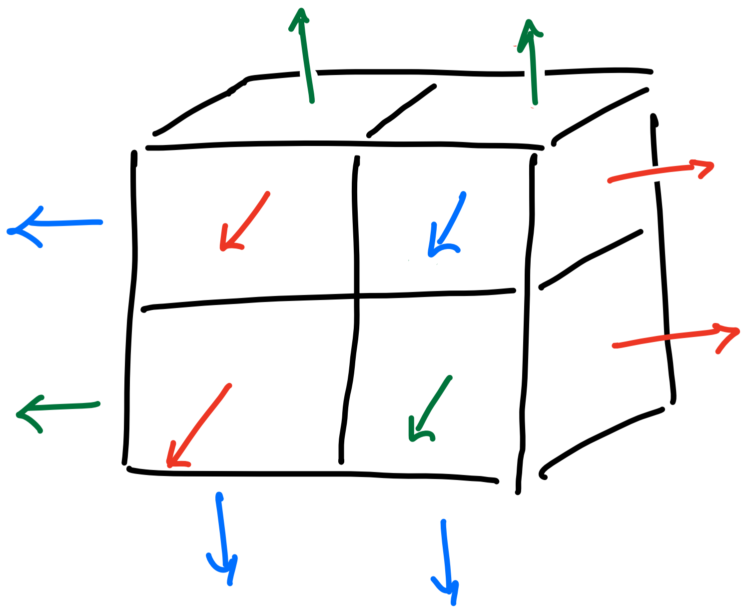


as soon as you specify z - you've specified then all. Right hand rule.



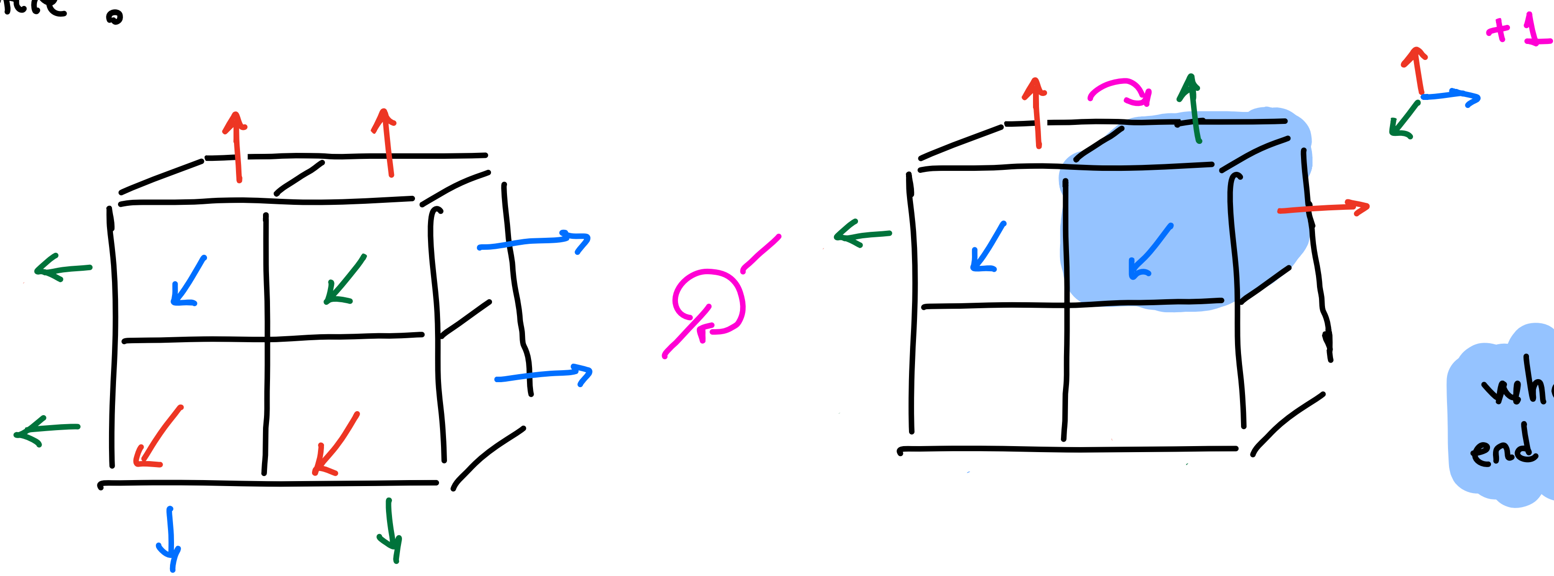
example:





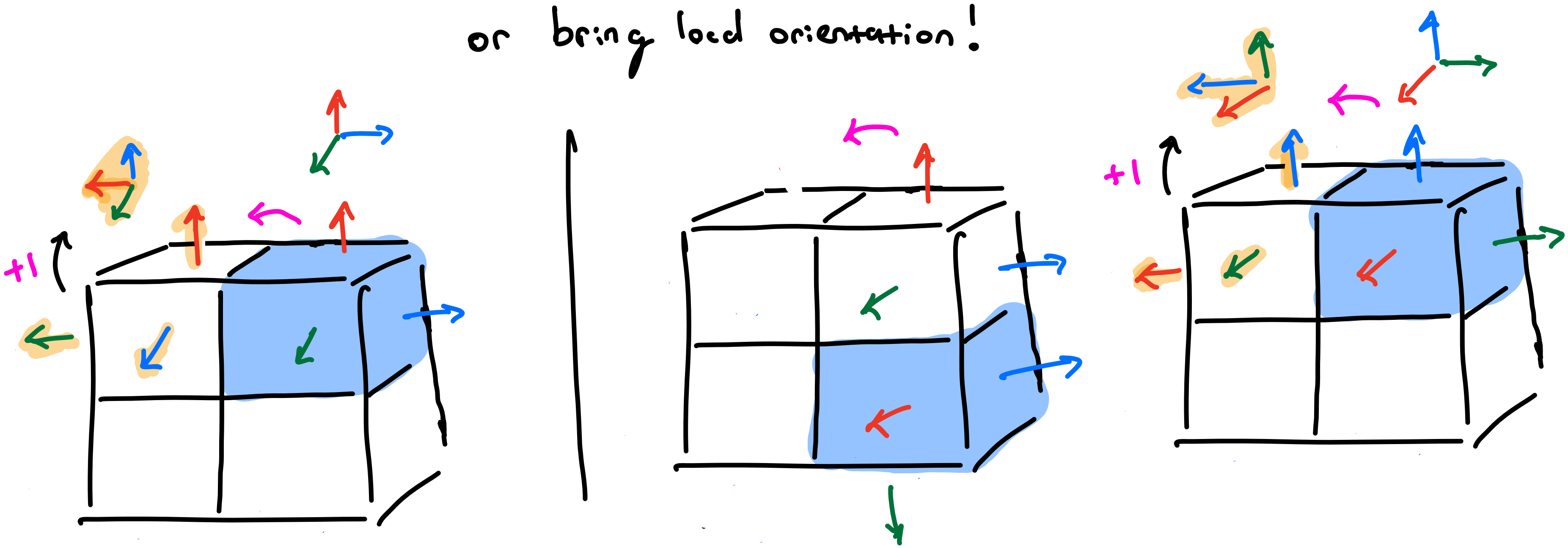
Sum of the local orientations = 3 or 0 in $\mathbb{Z}_3$.

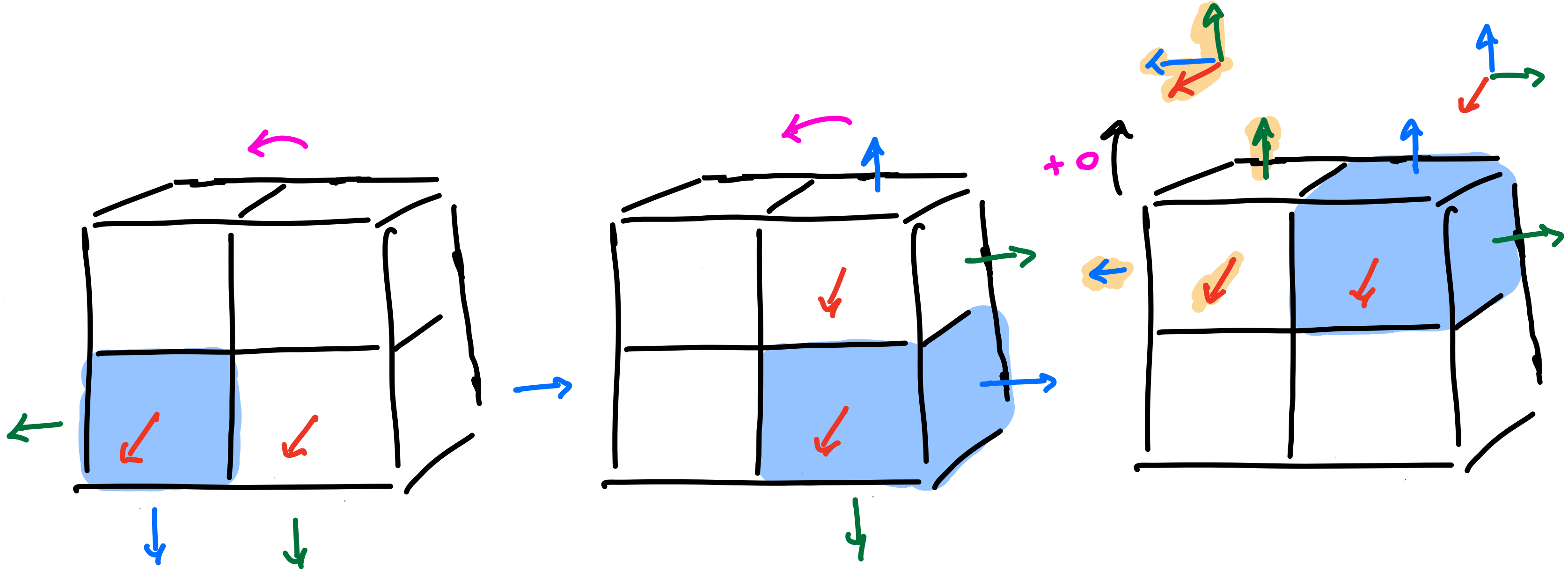
Picture:

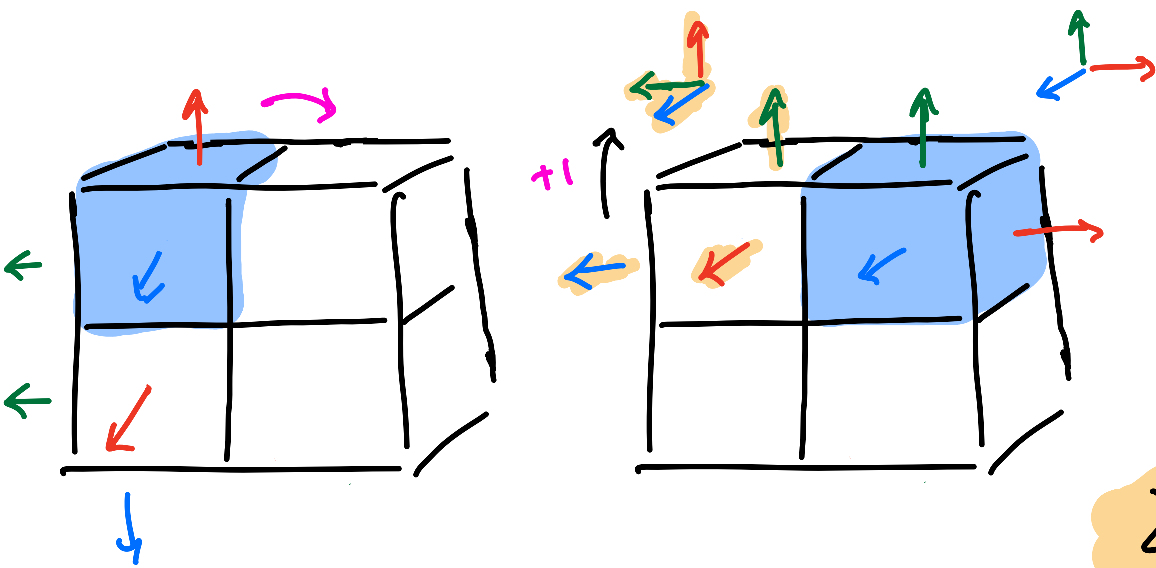


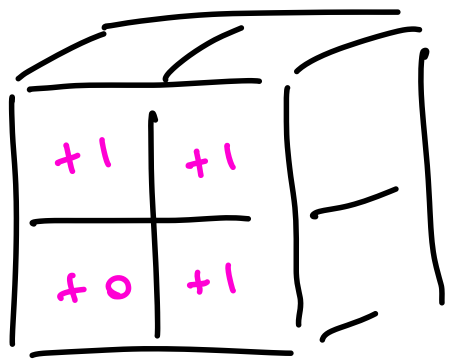
where you end up

or bring local orientation!





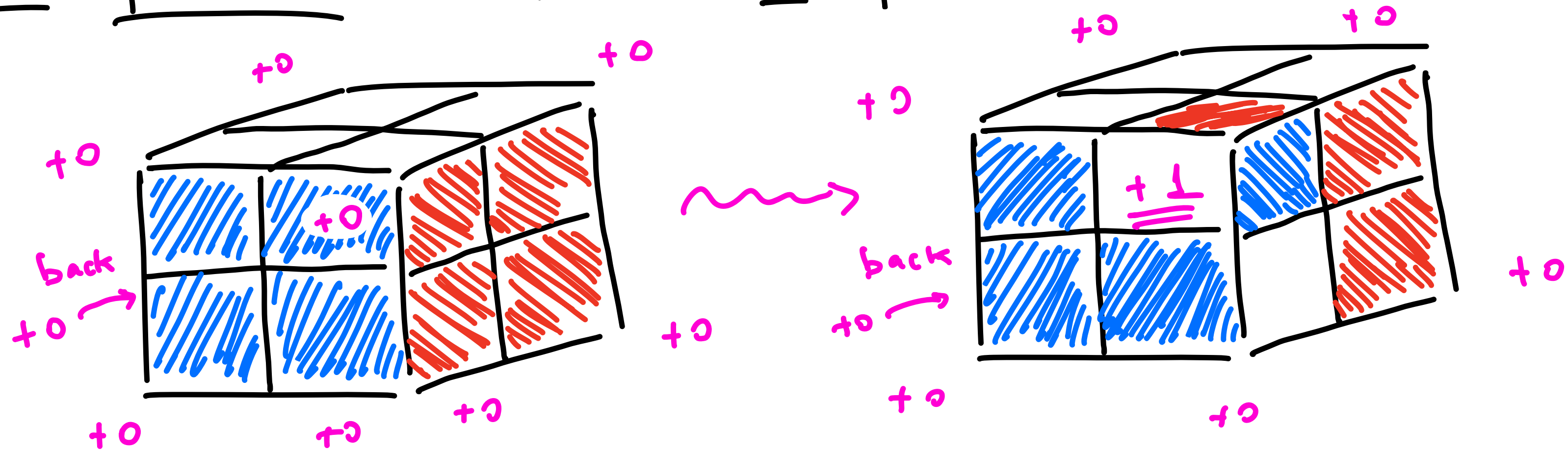




$\sum$ local orientations is preserved and equal to a multiple of 2π!

Upshot:

Not possible because does not preserve sum of local orientations

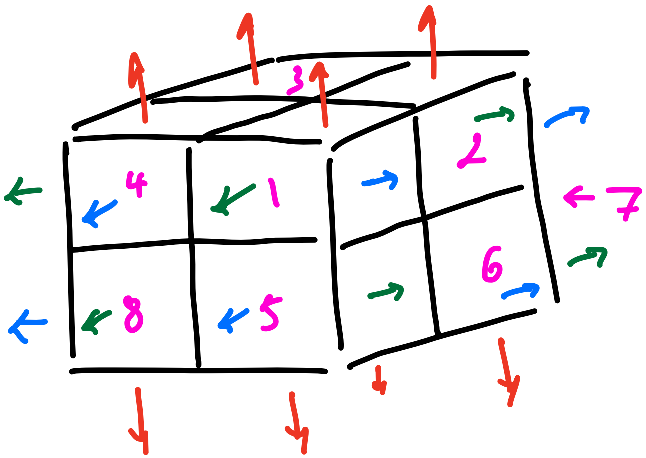


Fact: Pretty much everything else is legit. Specifically

$K \leq \mathbb{Z}_3^8$ is equal to $(x_1, x_2, \ldots x_8)$ so that

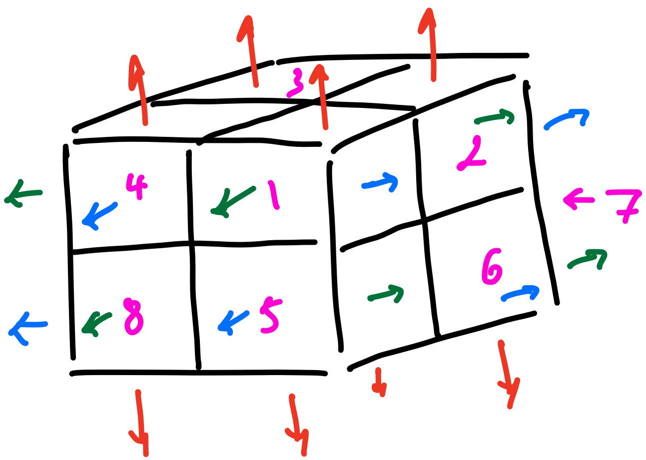
$x_1 + x_2 + \ldots + x_8 = \bar{0} \mod 3.$

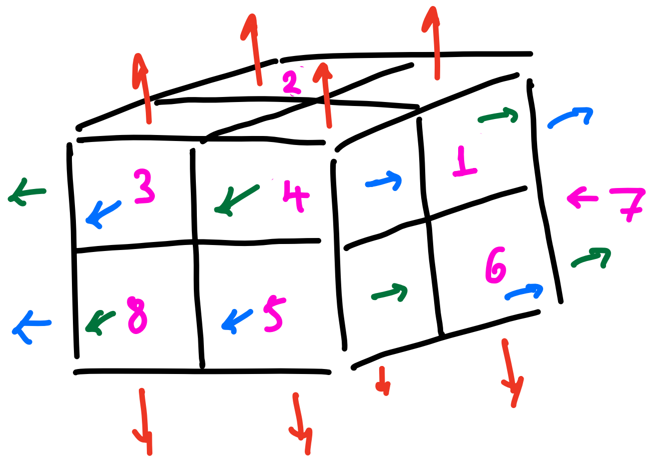
Special subgroup #2



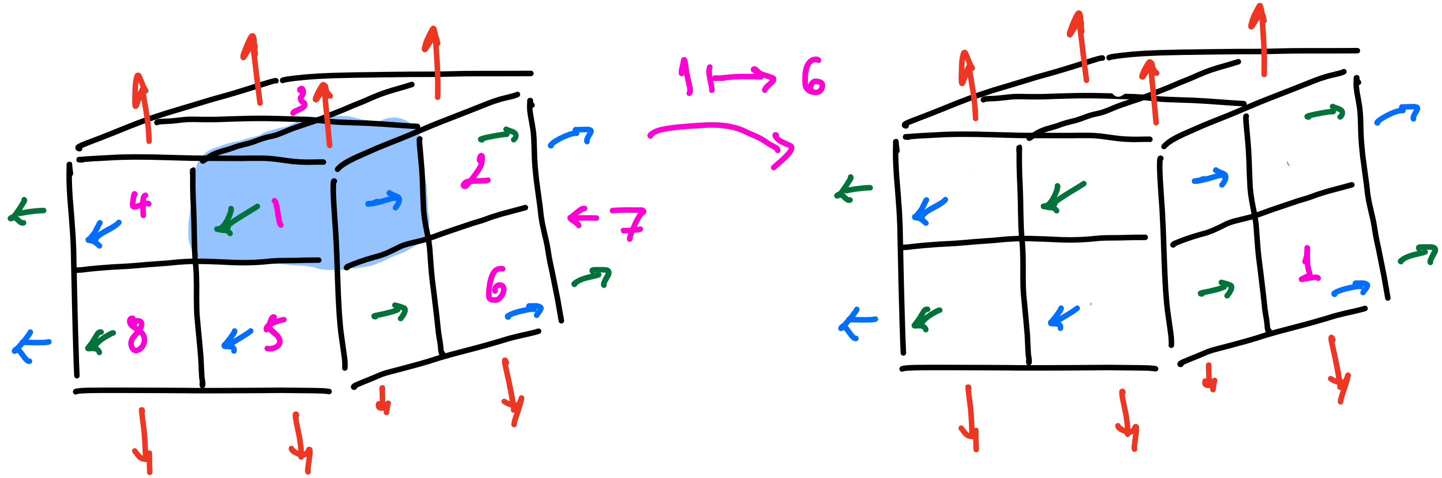
Pick your favorite local orientations on cube.

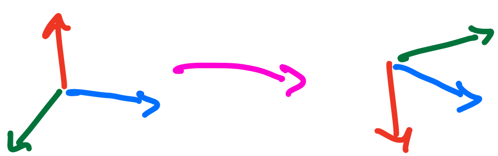
Let H = Subgroup of G that preserves this orientation system.





Observation: If you're in H, as soon as you say which cubes goes where you've specified the move.





Right-hand rule sorta deal.

Thus $H \hookrightarrow S_8$. **Fact:**

$$\begin{tikzcd} & H \arrow[d, hook, "inc"'] \arrow[dr, two heads, "\phi|_H"] & \\ K \arrow[r, hook, "inc"'] & G \arrow[r, two heads, "\phi"'] & S_8 \end{tikzcd}$$

onto

Putting pieces together

G = Group of symmetries of $2 \times 2 \times 2$

H = Group of (local) orientation preserving symmetries
(subgroup of symmetries S_8)

K = Group of symmetries that keep all cubes in place but mess around with local orientations
(subgroup of $\mathbb{Z}_3^8 = (x_1, x_2, \dots x_8)$ satisfying $x_1 + x_2 + \dots + x_8 = \bar{0} \bmod 3 \cong \mathbb{Z}_3^7$)

$H \cap K = 1$ (if you don't move cubes and preserve orientation, you didn't do much)

this means

$$\begin{tikzcd} H \arrow[d, hook, "\text{inc}"'] \arrow[dr, two heads, "\phi_H"] & \\ G \arrow[r, two heads, "\phi"'] & S_8 \end{tikzcd}$$

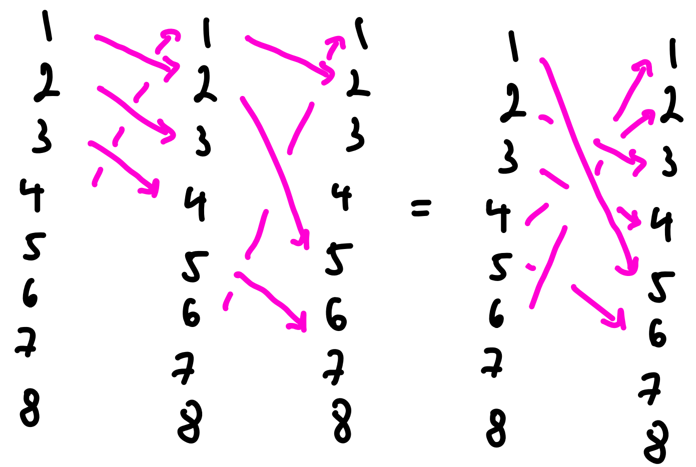
ϕ_H is injective (and surjective) which we call an **isomorphism**. Two groups admit an isomorphism called **isomorphic**. Denote by $\cong$. So $H \cong S_8$.

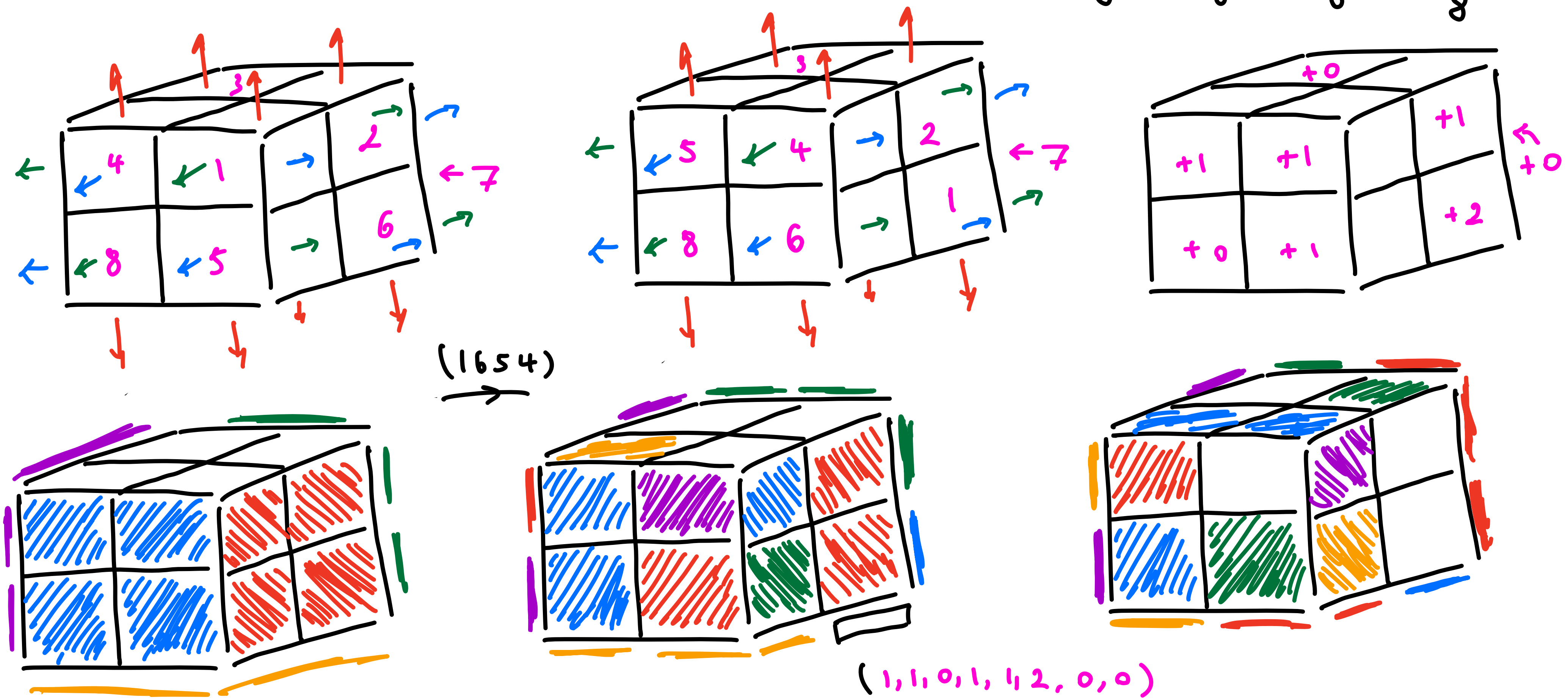
How does multiplication work?

In $K \cong \mathbb{Z}_3^7$ multiplication is understandable

$$\begin{bmatrix} +1 & +1 \\ +1 & +0 \end{bmatrix} \circ \begin{bmatrix} +2 & +1 \\ +0 & +0 \end{bmatrix} = \begin{bmatrix} +0 & +2 \\ +1 & +0 \end{bmatrix}$$

In H multiplication is understandable.





Group structure

thm: let $K \xhookrightarrow{inc} G \xrightarrow{\phi} Q$ so that there exists a $H \leq G$ satisfying $H \cap K = 1$ and $\phi|_H : H \twoheadrightarrow Q$ is an isomorphism.
Then every element $g \in G$ may be decomposed uniquely as $g = kh$ for $k \in K$ and $h \in H$.

Call G a semi-direct product of K and H.
Denoted by $G = K \rtimes_\phi H$.

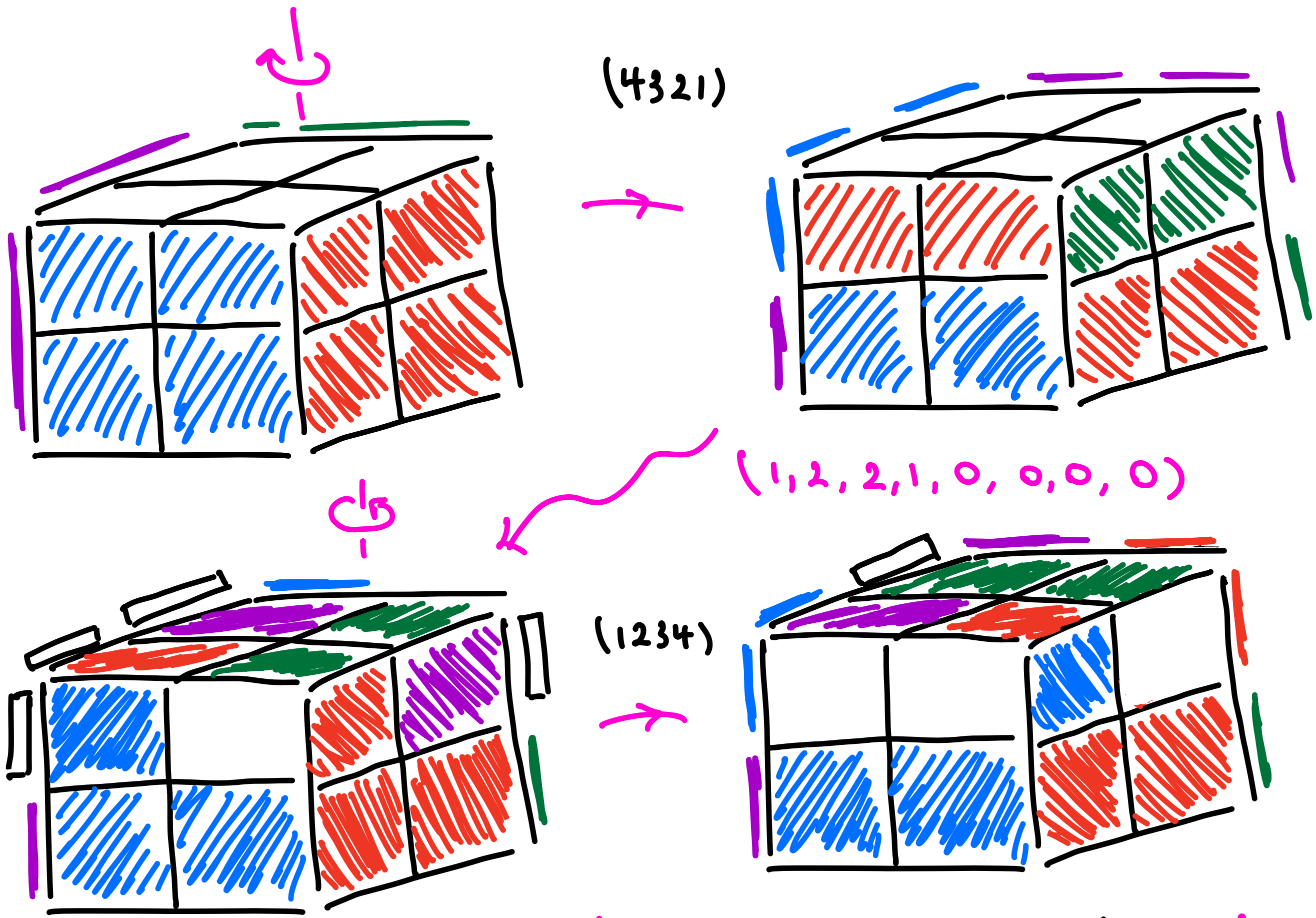
Example :

$$gg' = (kh)(k'h') = \big(k\underbrace{(hk'h^{-1})}_{\in K}\big)\overset{\in K}{}\big(hh'\big)^{\in H}$$

$$K \xhookrightarrow{inc} G \xrightarrow{\phi} S_8 = Q$$

K = orientation moving, cube preserving

H = orientation preserving, cube moving



(4321)

$(1,2,2,1,0,0,0,0)$

(1234)

$(1,1,2,2,0,0,0) = (1234)(1,2,2,1,0,0,0,0)(1234)^{-1}$

In general :

$$k = (1, 2, 0, 2, 0, 1, 2, 0)$$

$$h = (1\ 3\ 5)(2\ 4)(7\ 8)$$

$$(1, 2, 0, 2, 0, 1, 2, 0)$$

$$hkh^{-1} = (0, 2, 1, 2, 0, 1, 0, 2)$$

Note this induces an action of S_8 on K. That is for each $h \in S_8$, we get an isomorphism of K to itself (automorphism). The assignment of automorphisms respects group structure of S_8,

$$S_8 \xrightarrow{\varphi} \mathrm{Aut}(K) \quad \text{is a homomorphism}$$

$$h \longmapsto \{k \longmapsto \text{permute } k\}$$

Group structure:

$$G \cong K \rtimes_{\phi} S_8 \quad \text{where} \quad K \cong \mathbb{Z}_3^7$$

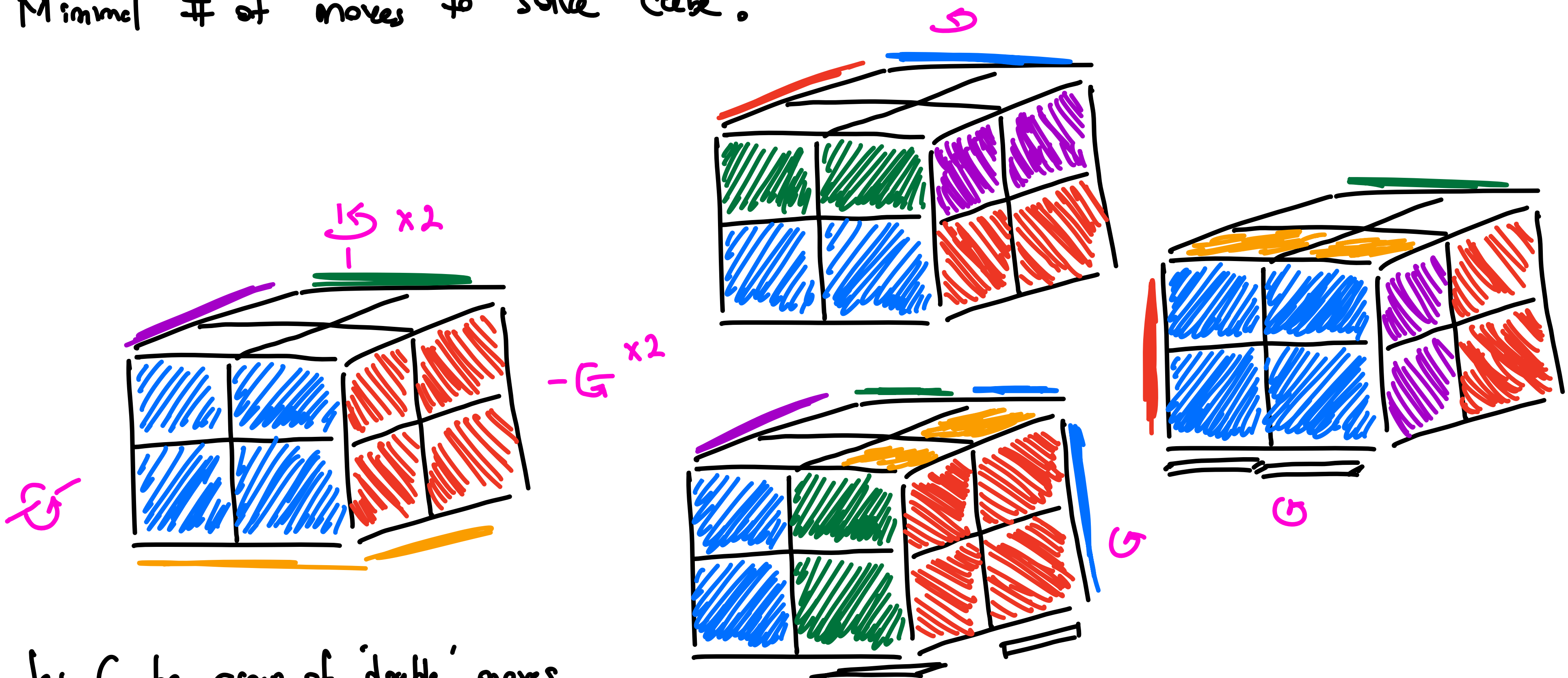
$$\text{so} \quad |G| = |K||S_8| = 3^7 \cdot 8!$$

$$= 88{,}179{,}840 \quad \text{possible arrangements}$$

so if you just mess around randomly you'll prolly never solve the thing.

Questions:

- How can you visualize a group so large?
- Does it have some other interpretation outside the puzzle?
- How to relate to $3 \times 3 \times 3$?
- What is subgroup structure?
- Minimal # of moves to solve cube?



Let G be group of 'double' moves.
Determine G upto isomorphism.

Thank You Folks!