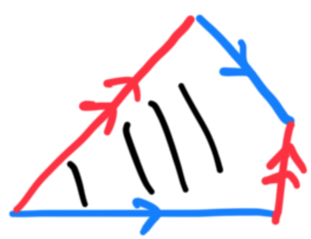
SYMPLECTIC TILINGS BILLIARDS

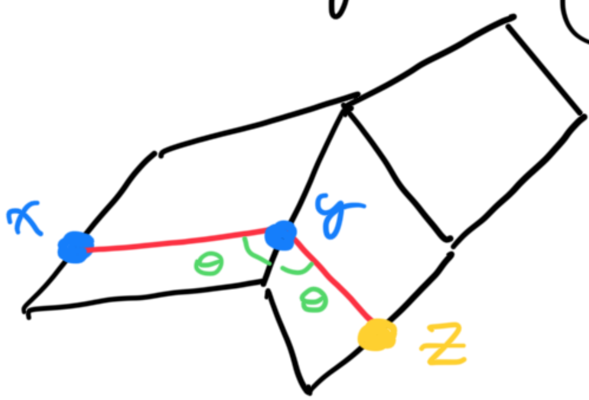
On 

(Charles Daly & Fabi Lander
Max Planck Institute in Leipzig)

13.03.26 - Universitat Autònoma de Barcelona

Union of two games

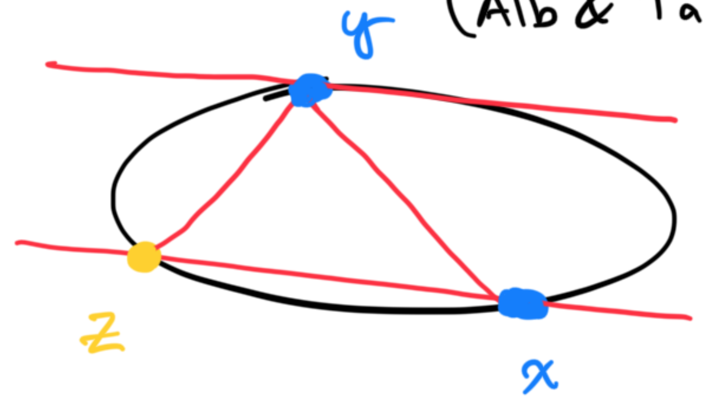
1. Tiling Billiards (Light?)



$xy \mapsto yz$

Angle (Similarity)

2. Symplectic Billiards (Alb & Tab '17)



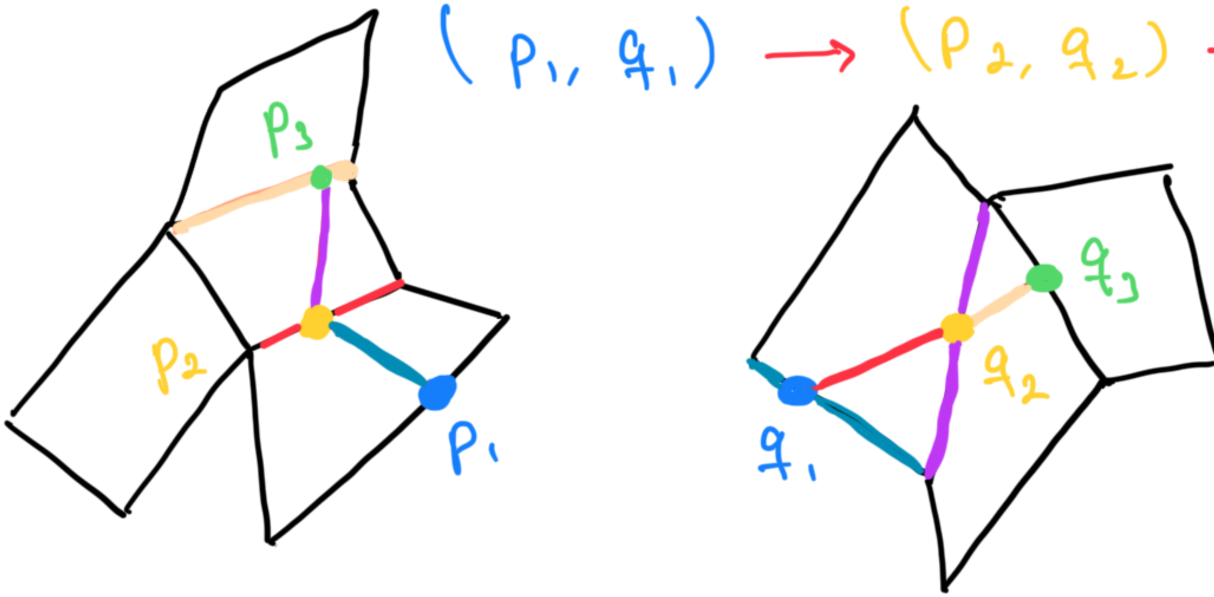
$xy \mapsto yz$

Parallel (Affine)

Symplectic Tiling Billiards

(Schwartz '23)

$(p_1, q_1) \to (p_2, q_2) \to (p_3, q_3)$



p_2: Draw line through p_1 $\parallel$ to q_1 edge

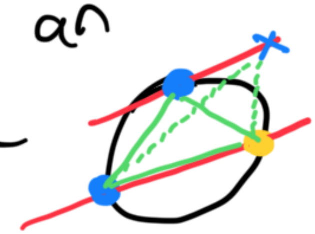
q_2: Draw line through q_1 $\parallel$ to p_2 edge

p_3: Draw line through p_2 $\parallel$ to q_2 edge

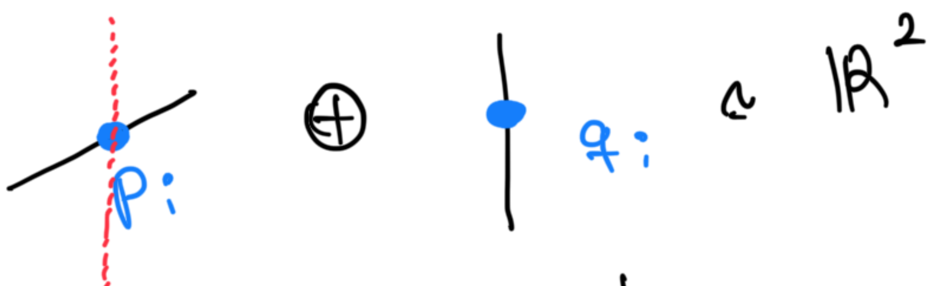
q_3: Draw line through q_2 $\parallel$ to p_3 edge

Some Remarks

- Called symplectic because there's an area form floating around somewhere



- Requires edges are transverse



- Breaks when hit vertex

Some More Remarks

The symmetries of the game are affine in nature as the game required only words like 'contained', 'edge', 'parallel'. Specifically if (T_A, T_B) are tilings, the dynamics are isomorphic to that played on (gT_A, gT_B)

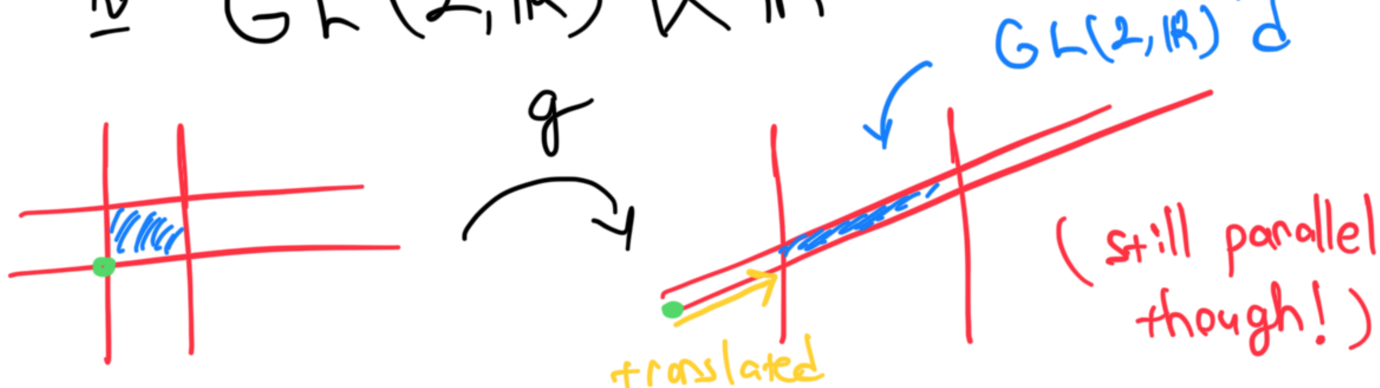
Two Dimensional Affine Geometry

$X = \mathbb{A}^2$ (the plane but forget origin)

$G = \mathrm{Aff}(2,\mathbb{R})$ group of translations

$\mathbb{R}^2$ + General linear group

$\cong GL(2,\mathbb{R}) \ltimes \mathbb{R}^2$



Affine Symmetry

	$G = \mathrm{Isom}(\mathbb{R}^2)$	$G = \mathrm{Aff}(2, \mathbb{R})$
length	✓	✗
angle	✓	✗
area	✓	✗
orthogonal proj	✓	✗
parallelism	✓	✓
convexity	✓	✓
barycenters	✓	✓

Affine structures on surfaces

__def:__ An affine structure on a surface S is an atlas of charts $\phi_\alpha : U_\alpha \subset S \to A^2$ so that coordinate changes are locally restrictions of affine transformations.

Observations

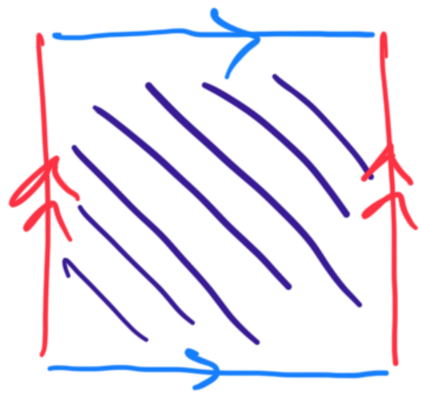
Given such a structure on S, the surface inherits the local geometry of $\mathbb{A}^2$. One can define notions of lines (geodesics), parallel, convexity, etc.

(Complete) Affine Surfaces

<u>def</u>: An affine surface is called <u>complete</u> if any geodesic can be extended indefinitely in both directions for all time.

<u>thm</u>: An affine surface is complete iff it is quotient $\mathbb{A}^2/\Gamma$ where Γ is discrete subgroup of affine transformations acting properly discontinuously and freely

Examples of Complete & Incomplete



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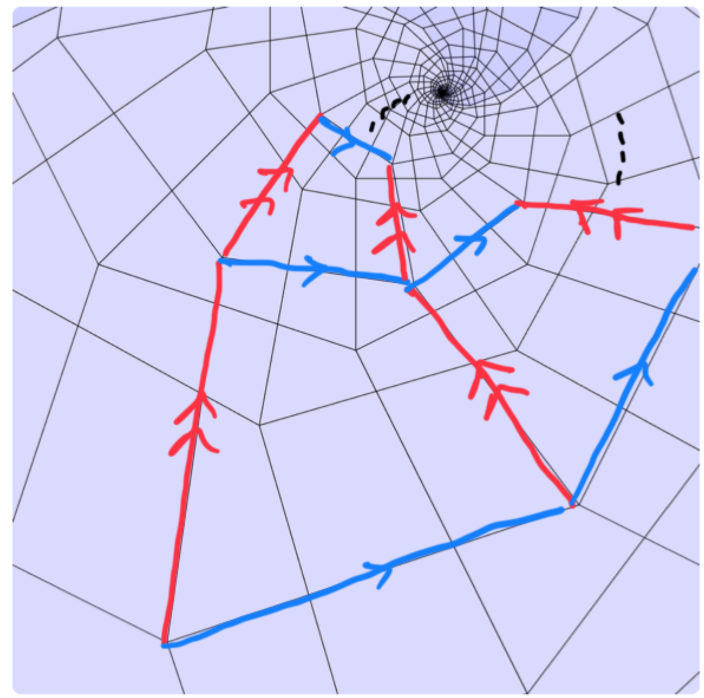


$\mathbb{R}^2 / \langle \text{two translations} \rangle$

quadrilateral with edges identified by two commuting similarity transformations

Classification of Complete Affine Structures

(Baues & Goldman '05
Goldman '25)

T^2

or

K

<u>theorem</u>: The deformation space of <u>complete</u> affine structures on T^2 is homeomorphic to the plane. The mapping class group acts by the standard $SL(2,\mathbb{Z})$-action on $\mathbb{R}^2$ where the fixed origin corresponds to the Euclidean structure.

Classification continued

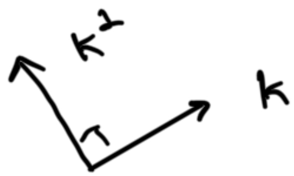
Every complete affine torus determined by single vector $\vec{k} = (k_1, k_2) \in \mathbb{R}^2 \cong$ Deformation space of complete affine tori

$\pi_1\Big($  $\Big) \xhookrightarrow{\;hol_{\vec{k}}\;} \mathrm{Aff}(2, \mathbb{R})$

$$\underbrace{n\alpha + m\beta}_{\gamma} \longmapsto \begin{aligned} \mathrm{lin}(\gamma)\,\vec{v} &= \vec{v} + (\gamma \cdot \vec{k})(\vec{v} \cdot \vec{k})(-\vec{k})^{\perp} \\ \mathrm{trans}(\gamma) &= \gamma + \tfrac{1}{2}(\gamma \cdot \vec{k})^2 (-\vec{k})^{\perp} \end{aligned}$$

Holonomy Observations

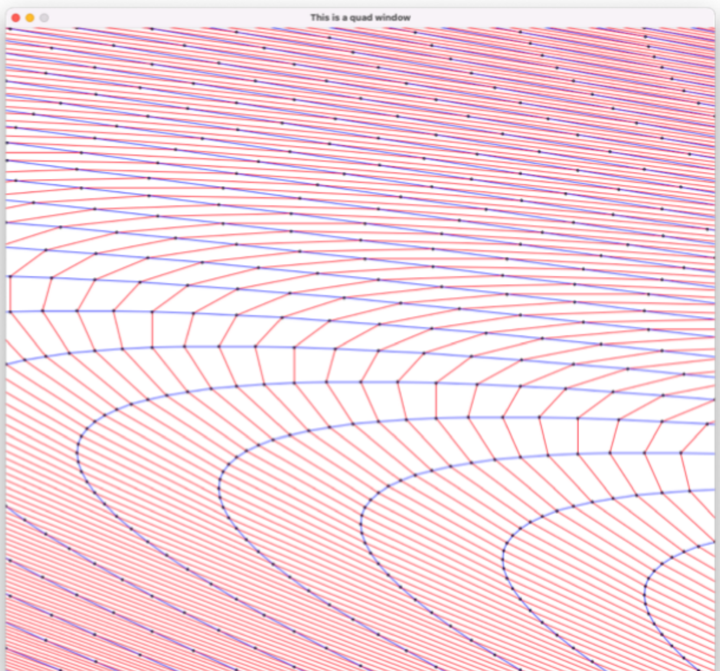
- Invariant vector field generated by $k^\perp$ on T^2 can always go in that direction. To the right vs left also makes sense, but not angle.

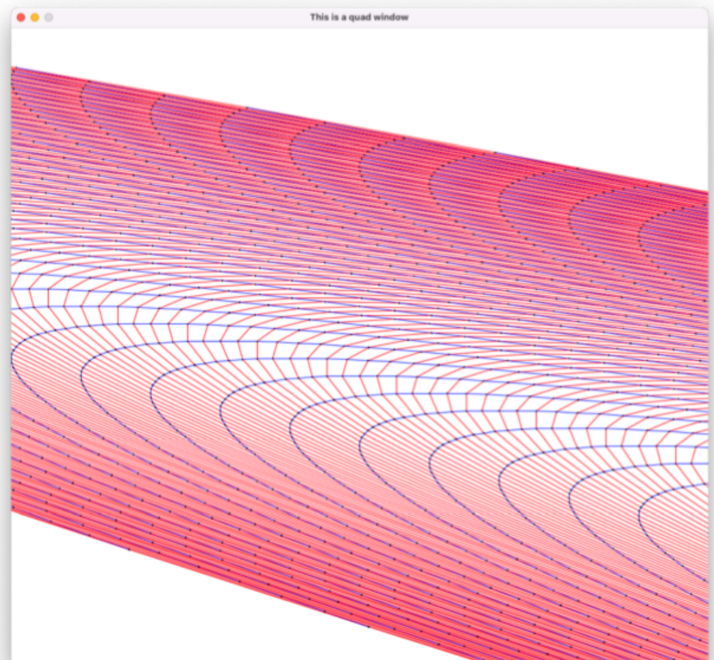


- Invariant one form $\eta(\vec{v}) := \langle \vec{v}, \vec{k} \rangle$
  leaves generated by flow of invariant vector field.
- If $\vec{k} \in \mathbb{Q}^2$, then holonomy contains pure translations.
  Construct convex quadrilateral fundamental domains!

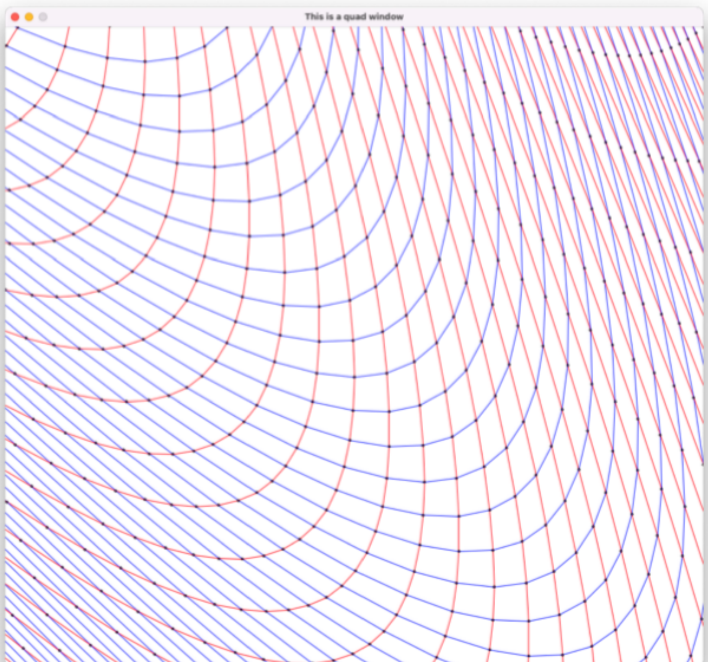
$$\mathrm{lin}(\gamma)\,\vec{v} = \vec{v} + (\gamma \cdot \vec{k})(\vec{v} \cdot \vec{k})(-\vec{k})^\perp$$

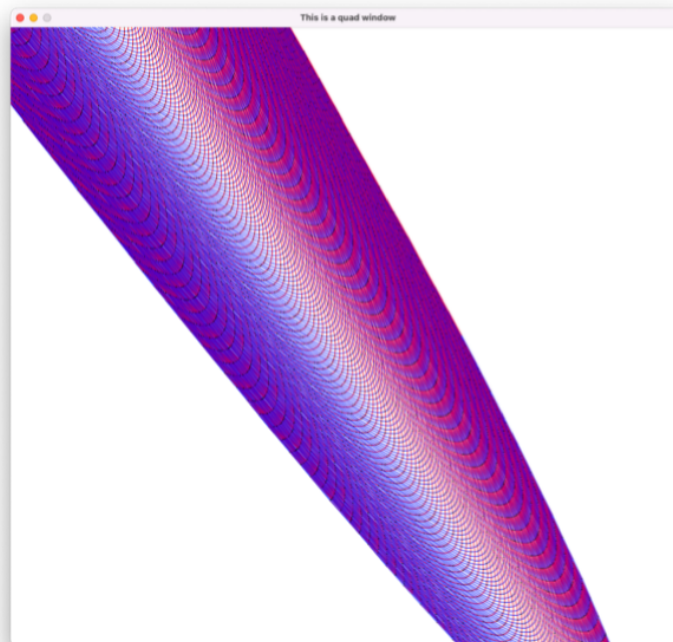
Some Examples $(k_1, k_2) = (1/4, 1)$





Some Examples $(k_1, k_2) = (-3/5, -2/5)$





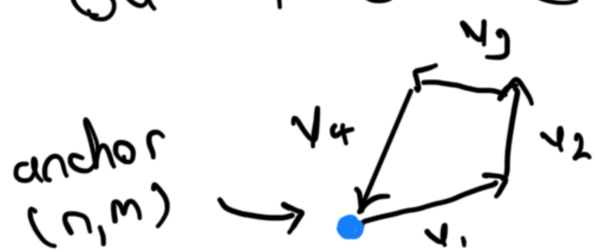
One more example ....

Käthe-Kollwitz-Straße 66, 04109, Leipzig

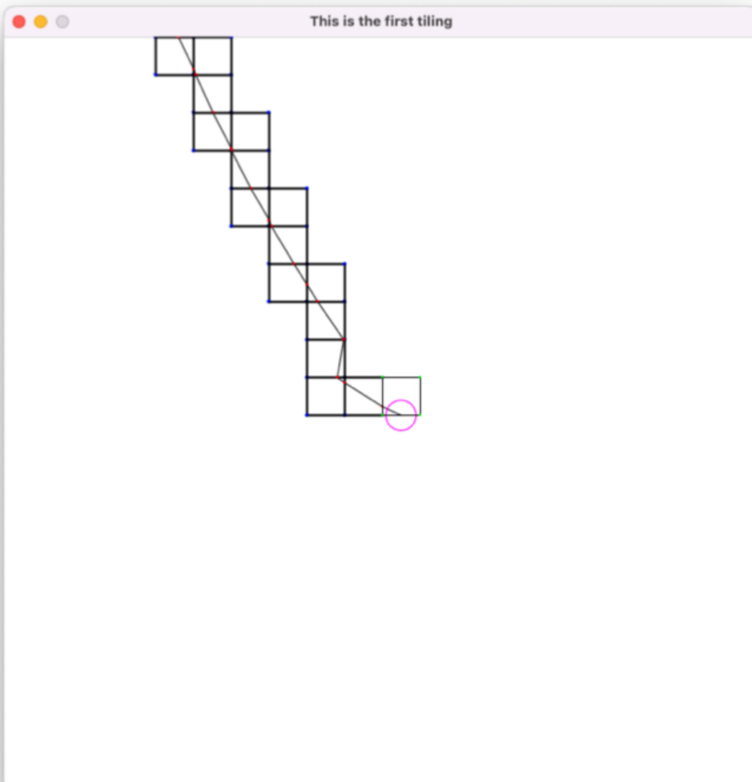
Playing on rational tori:

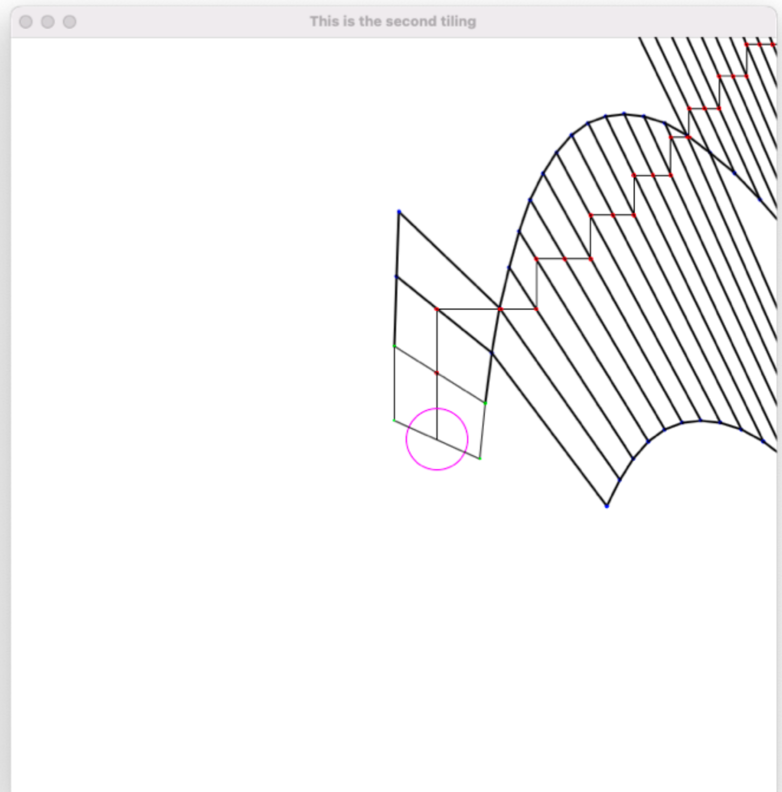
- Let one tile be Euclidean
- For other tile, assume $k_1, k_2 \in (\mathbb{Q}^*)^2$

  (This guarantees transversality in long-run)
- Orient our tiles counter clockwise



Some Orbits $(k_1, k_2) = (1, 1/4)$





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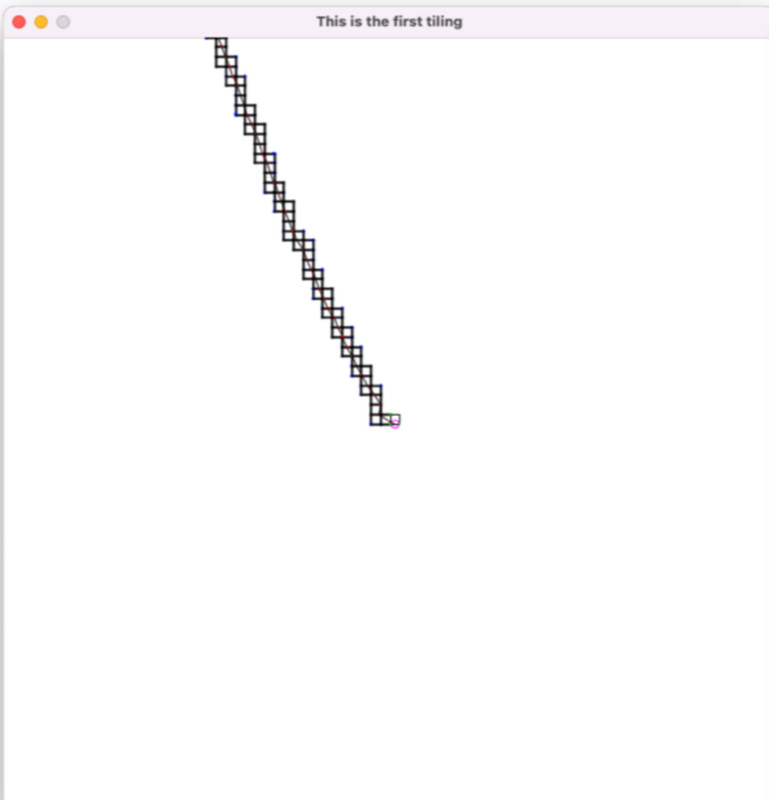
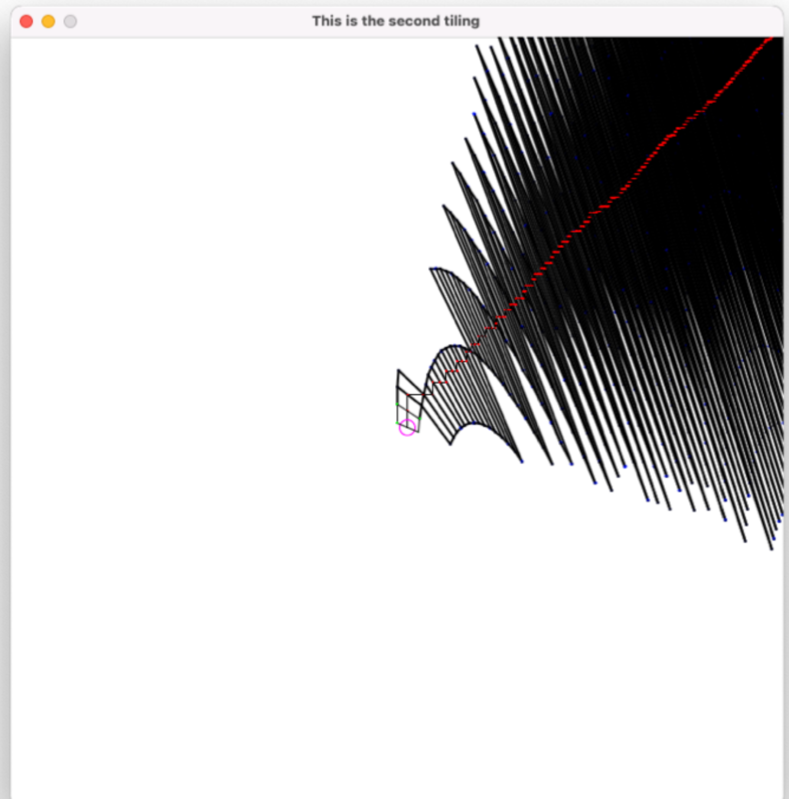
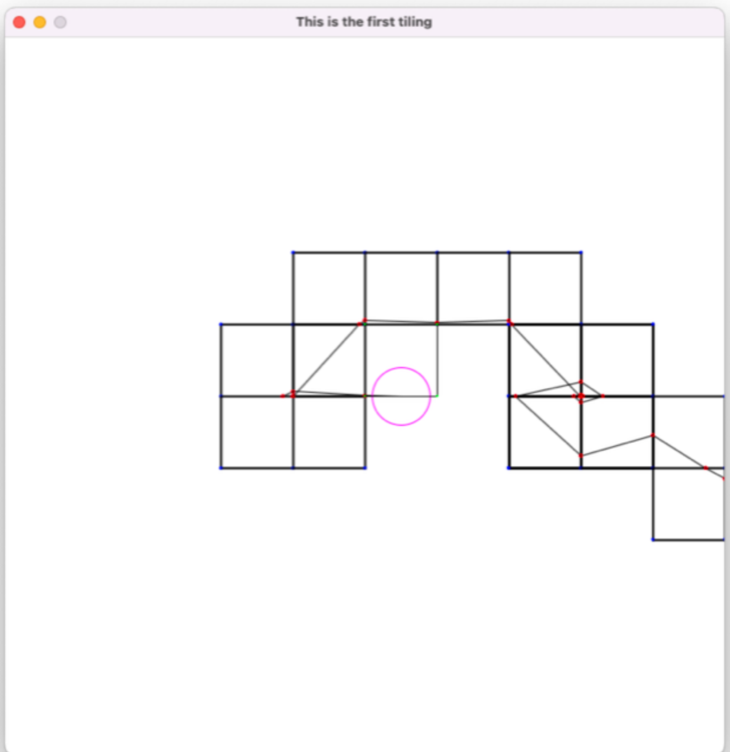
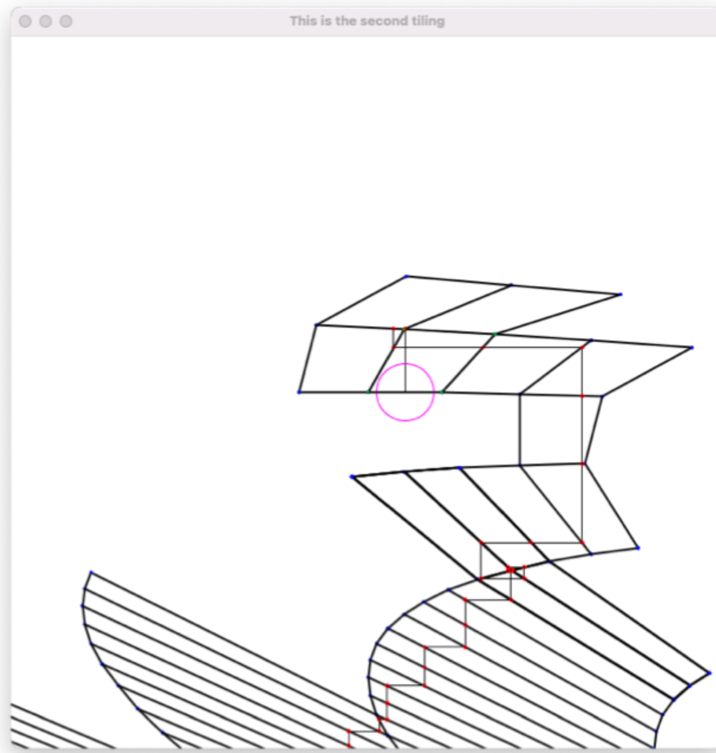


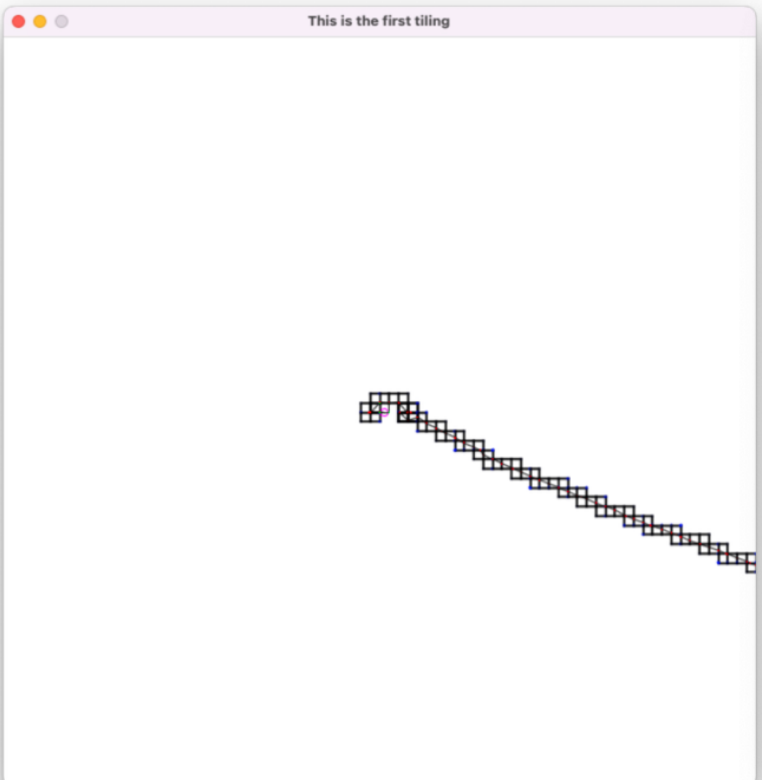


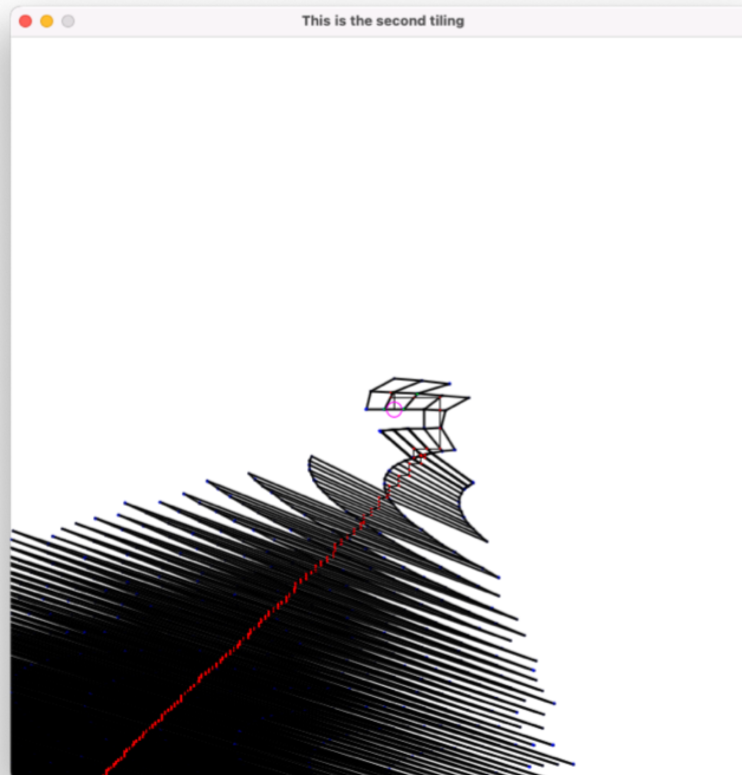
Some orbits $(k_1, k_2) = (1/4, 1)$



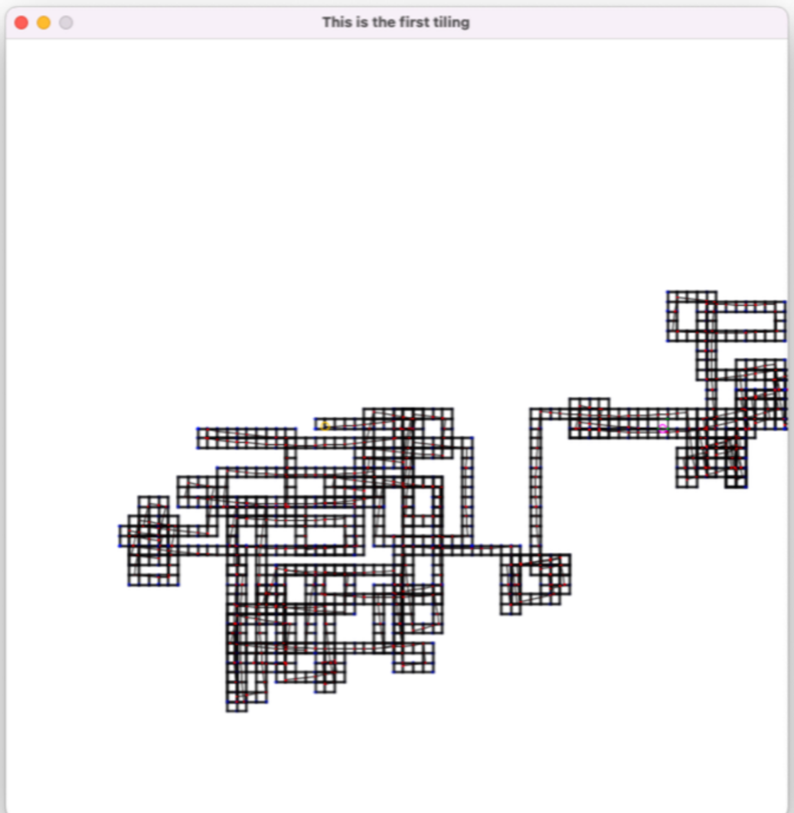


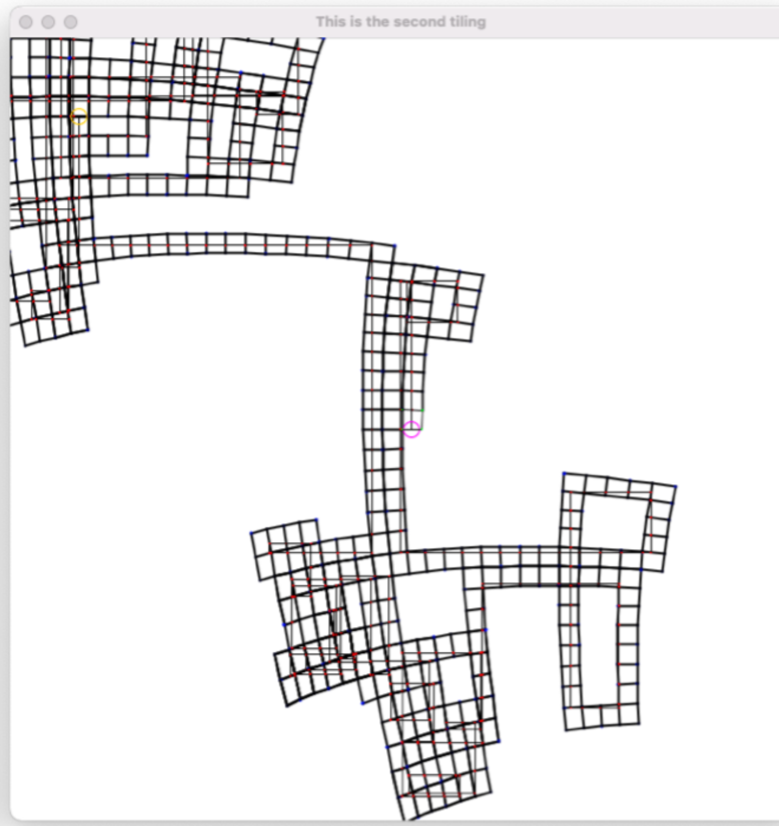
Some Orbits $(k_1, k_2) = (1/4, 1)$





Some orbits $(k_1, k_2) = (1/4, 1/4)$





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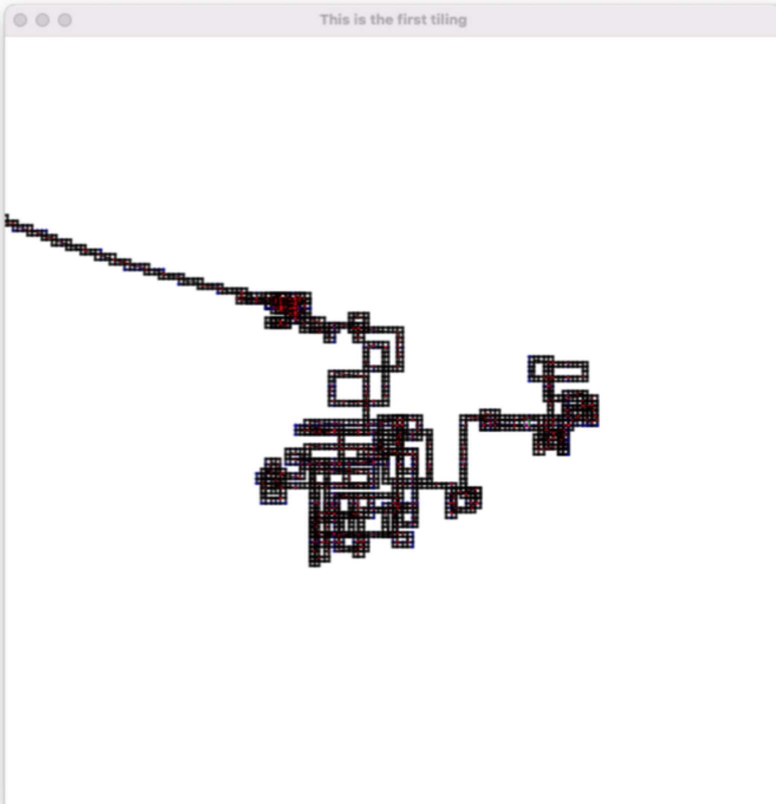
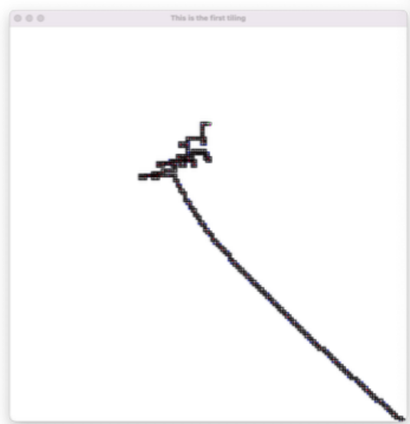


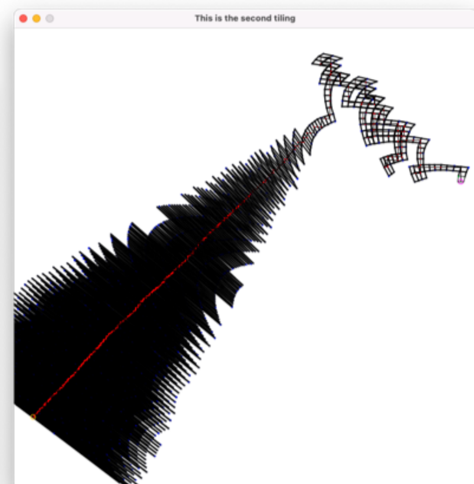
Conclusion :

Seem to generically diverge provided $\vec{k} \neq \vec{0}$. However, does not seem to be a way to determine after how many iterates. But the idea is clear for sufficiently 'far away' non-Euclidean tiles

Theorem (DL: Work in Progress '26)

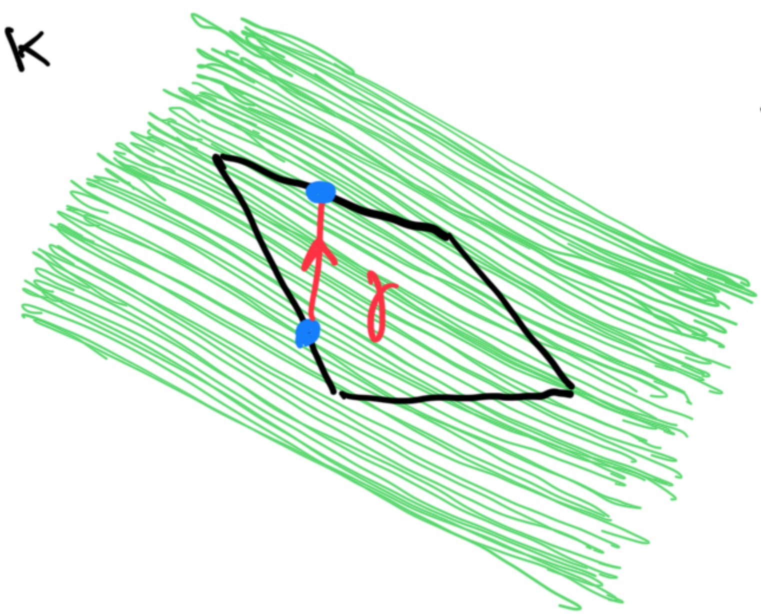
For any $\vec{k} \neq 0 \in (\mathbb{Q}^\times)^2$, there exists a sufficiently large $C_{\vec{k}} > 0$ so that any tile anchored at (n,m) with $(n,m) \cdot k > C_k$ admits divergent orbits.





Way to Measure Divergence

To each orbit, can assign a # which measures how far along $\vec{K}$ moving.

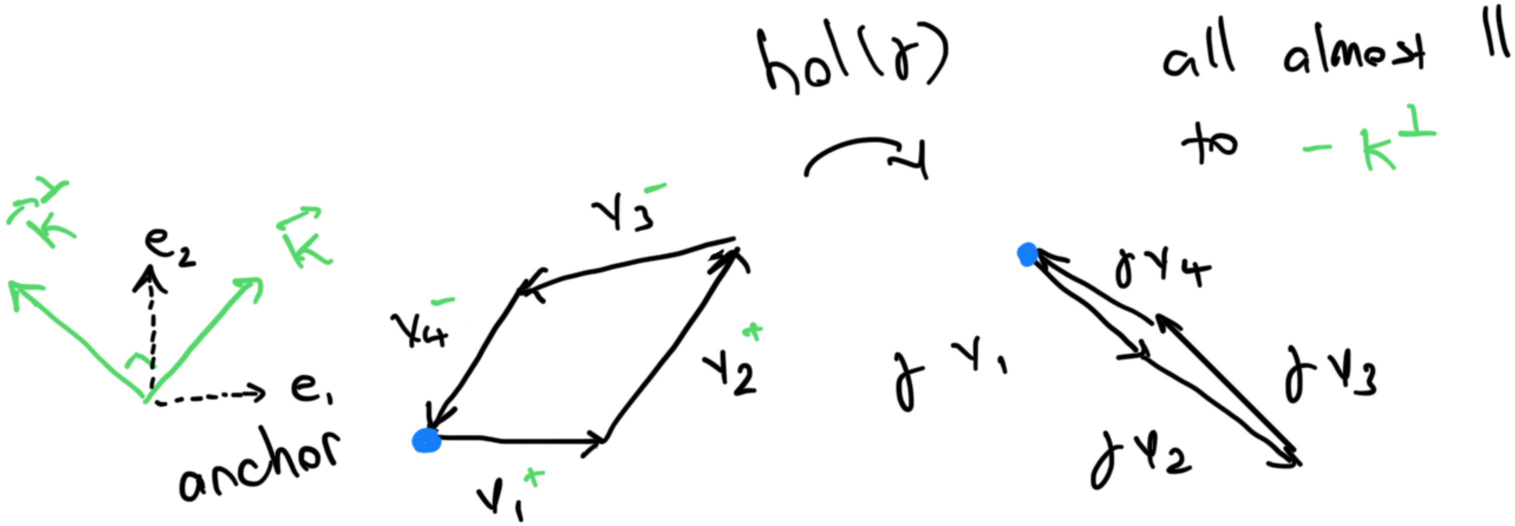


$$\int_\gamma \omega > 0$$

(oriented) foliation by leaves of ω.

Key Observation

As $\gamma = (n, m)$ becomes more parallel to $\vec{K}$, the tiles flatten out parallel to $K^\perp$.



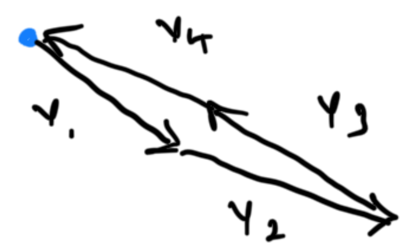
Why Tiles Flatten

Recall $\mathrm{hol}(\gamma)\,\vec{v} = \vec{v} + (\gamma \cdot \vec{K})(\vec{v} \cdot \vec{K})(-K^{\perp})$

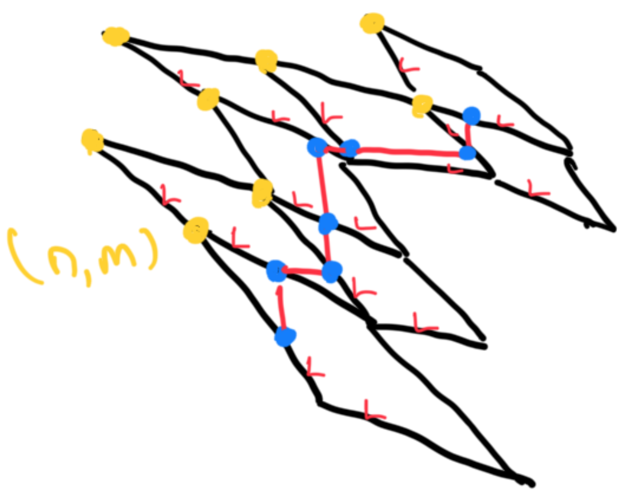
So when $(\gamma \cdot \vec{K}) >> 0$ we have

$\mathrm{hol}(\gamma)\,\vec{v}$ is approximately parallel to

$\langle K^{\perp} \rangle$.



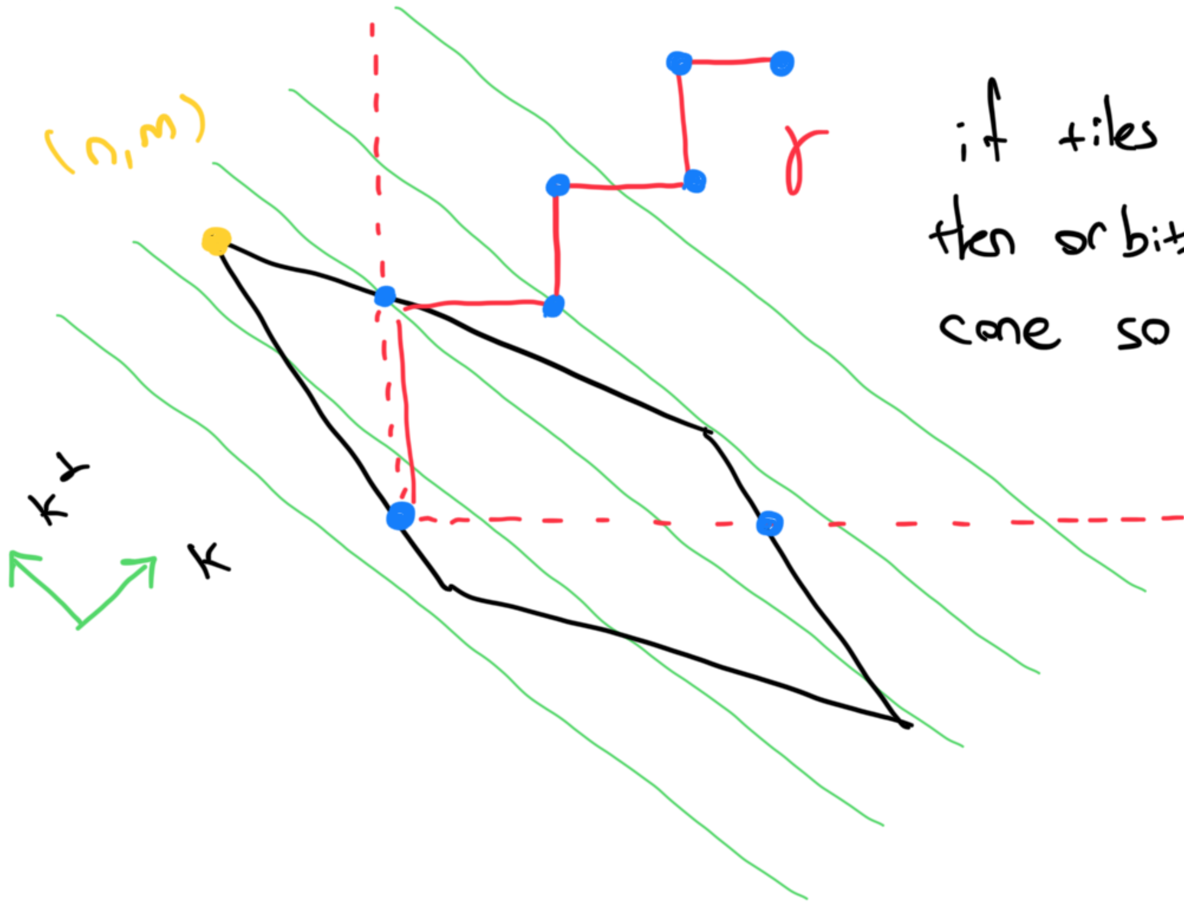
Move through Flat Tiles



if tiles remain flat,
then can only move in one of two directions

Anchors • (n,m) determine how flat a tile is.

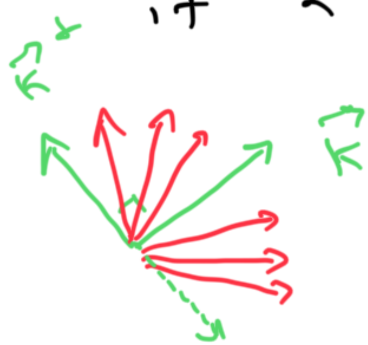
Want to Guarantee Tiles Remain Flat



if tiles remain flat then orbits remain in cone so $\int_\gamma \omega \to \infty$

Positive / Negative Directions

def: Say a vector $\vec{v}$ is positive/negative if $\langle \vec{v}, \vec{k} \rangle$ positive/negative.

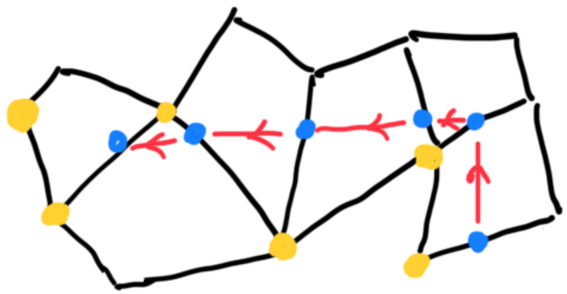


fan of positive directions

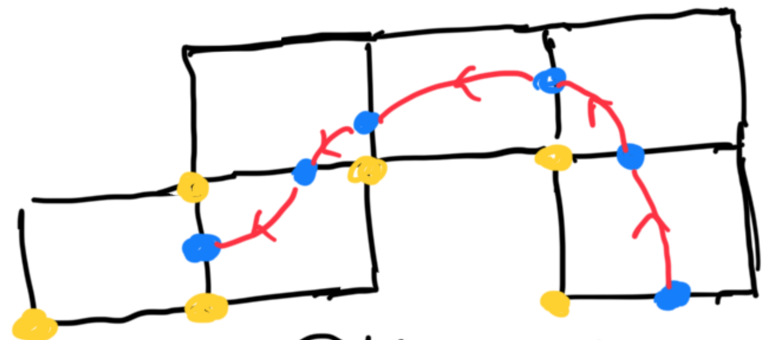
$$\langle \mathrm{hol}_\gamma \vec{v}, \vec{k} \rangle = \langle \vec{v}, \vec{k} \rangle$$

A loop $\gamma \in \pi_1(T^2)$ is positive iff $\langle (n,m), k \rangle$ is positive where $\gamma = n\alpha + m\beta$.

Orbits to Loops



in $\mathbb{A}^2$

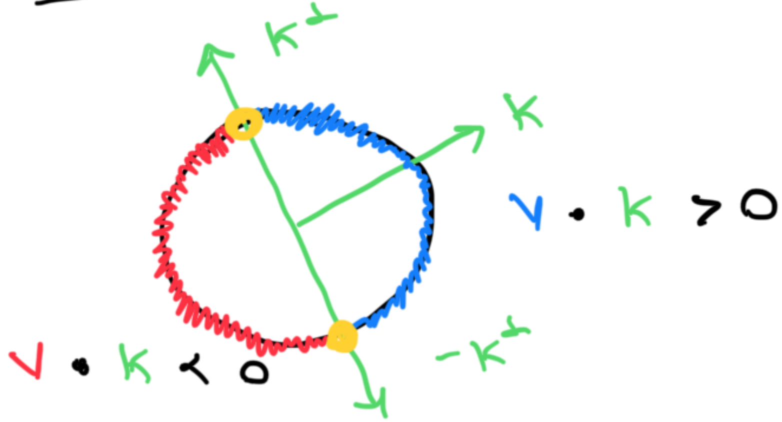


in $\widetilde{T}^2 \simeq \mathbb{R}^2$

Assign orbit holonomy

$$e_2 - e_1 - e_1 - e_2 - e_1 = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

Parabolic Holonomy



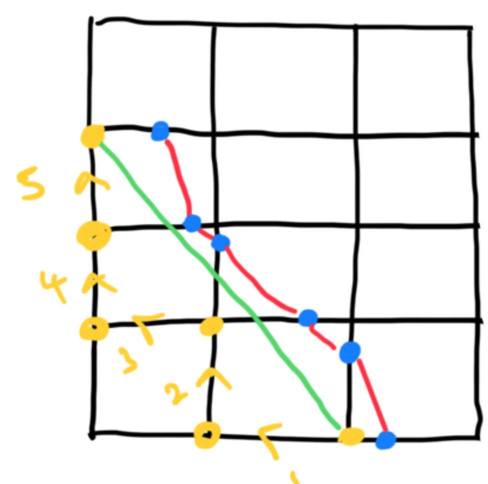
positive/negative directions preserved for all $\gamma \in \pi_1(T^2)$

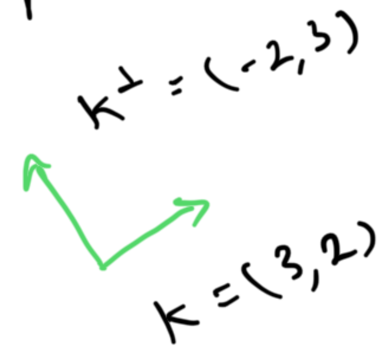
Positive loops move positive directions towards $-k^\perp$ and negative directions towards $k^\perp$.

Control on Tile Shape

if $\vec{k} = \left(\frac{a}{c}, \frac{b}{c}\right)$ then after $|a| + |b|$ moves, at worst, a tile will get translated if move in only positive directions.

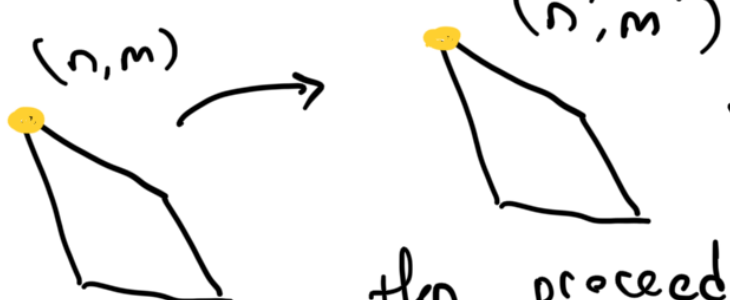
orbit gets assigned loop $\gamma = (-2, 3)$





Tiles Must Flatten

If picked a tile at (n,m) that stays sufficiently flat for $|a|+|b|$-moves, then at worst, after these moves the tile



translated

then proceed argument again.

Because $\int_\gamma \omega \to \infty$, tiles must flatten.

Future Directions

- Try and show this behavior is generic i.e. the measure of configurations which do not diverge is trivial
- Implement this for $k \in \mathbb{Q}(\sqrt{2})^2$. All of this is affine! Only algebraic operations are $+, -, \cdot, \div$.
- Better understand the case where $\vec{k}_1 \neq 0$ and $\vec{k}_2 \neq 0$

Thank you for your time!